Inverse Limits with Set Valued Functions

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Aknowledgement

This work is based on the paper :
W.T. Ingram and W.S. Mahavier, Inverse limits of upper semi-continuous set valued functions, Houston J. Math. 32 (2006), 119-130.

Upper semi-continuous set valued functions.

- All spaces are separable and metric. A continuum is a compact and connected space.
- If X is a compact space, then $f : X \to 2^Y$ is usc if and only if $G = \{(x, y) | y \in f(x)\}$ is a closed subset of $X \times Y$.
- If X is a compact space and $f : X \to Y$ is continuous, then $f^{-1} : Y \to 2^X$ is usc.
- If X, Y, and Z are compact spaces and $f : X \to 2^Y$ is usc, and $g : Y \to 2^Z$ is usc, and if $g \circ f = \{(x, z) | \exists y \in f(x) \ni z \in g(y)\}$, then $g \circ f : X \to 2^Z$ is usc.

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- If X is compact and $f : X \to 2^X$ then $\varprojlim f = \{(x_1, x_2, x_3, \ldots) | \forall i \; x_i \in f(x_{i+1})\}$.

  Where $(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X$

  with metric $d(x, y) = \sum_{i=1}^{\infty} \frac{d_X(x_i, y_i)}{2^i}$.

- The shift map $\sigma : \prod_{i=1}^{\infty} X \to \prod_{i=1}^{\infty} X$ is defined $\sigma((x_1, x_2, x_3, \ldots)) = (x_2, x_3, x_4, \ldots)$.
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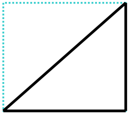
Example.

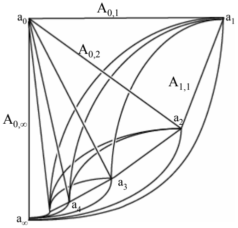


$G \subset [0,1]\times[0,1]$

$$A_{i,j} = \{(\underbrace{0,0,\ldots,0}_{i},\underbrace{x,x,\ldots,x}_{j},1,1,\ldots) \mid 0 \le x \le 1\}$$

$$a_i = (\underbrace{0,0,\ldots,0}_{i},1,1,\ldots)$$

$$\lim_{\leftarrow} G = \bigcup_{i=0}^{\infty}\bigcup_{j=0}^{[\infty]} A_{i,j} \approx \text{ a complete graph with vertices } \{a_i\}$$



Crossovers

Suppose $x, y \in M \subset \prod_{j=1}^{\infty} X_j$ and $x_i = c = y_i$ for some i.

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$y = (y_1, y_2, y_3, \ldots, c, y_{i+1}, y_{i+2}, \ldots)$

Define $Cr_i(x, y) = (x_1, x_2, x_3, \ldots, c, y_{i+1}, y_{i+2}, \ldots)$.

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then $f_i = \pi_i \circ \pi_{i+1}^{-1}$.

So if $\varprojlim(f_i, X_i)$ is compact, then each f_i is usc.

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Suppose $M \subset \prod_{i=1}^{\infty} X_i$ is compact and $X_i = \pi_i(M)$ for each i, and define $f_i : x_{i+1} \to 2^{X_i}$ by $f_i = \pi_i \circ \pi_{i+1}^{-1}$

Then $M = \varprojlim(f_i, X_i)$ if and only if $Cr(M) = M$.

Proof:

$M = \varprojlim(f_i, X_i) \Longrightarrow Cr(M) = M$ is easy.

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Assume $Cr(M) = M$ and let $x = (x_1, x_2, x_3, \dots) \in \varprojlim(f_i, X_i)$.

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$\Longrightarrow x_j \in \pi_i(\pi_{i+1}^{-1}(x_{j+1}))$ for each j.
$\Longrightarrow$ for each j there is $(c_1, c_2, \dots, x_j, x_{j+1}, c_{j+2}, c_{j+2}, \dots) \in M$.
$Cr(M) = M$
$\Longrightarrow$ for each j there is $x^j = (x_1, x_2, \dots, x_j, x_{j+1}, c_{j+2}, c_{j+2}, \dots) \in M$.
But M is compact and $\{x^j\} \rightarrow x$, so $x \in M$.
$\Longrightarrow \varprojlim(f_i, X_i) \subset M$.

Suppose $M \subset \prod_{i=1}^{\infty} X_i$ is compact and $X_i = \pi_i(M)$ for each i, and define $f_i : x_{i+1} \to 2^{X_i}$ by $f_i = \pi_i \circ \pi_{i+1}^{-1}$
Then $M = lim(f_i, X_i)$ if and only if $Cr(M) = M$.

Proof:
$M = \underleftarrow{lim}(f_i, X_i) \implies Cr(M) = M$ is easy.
$M \subset \underleftarrow{lim}(f_i, X_i)$ is easy and depends only on $f_i = \pi_i \circ \pi_{i+1}^{-1}$.
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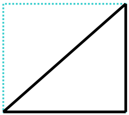
But M is compact and $\{x^j\} \to x$, so $x \in M$.
$\implies \underleftarrow{lim}(f_i, X_i) \subset M$.

Theorem

If X *is compact and* $M \subset \prod_{i=1}^{\infty} X$ *is compact, then there is a set* $G \subset X \times X$ *such that* $M = \varprojlim G$ *if and only if* $Cr(M) = M$ *and* $\sigma(M) = M$.

Example.

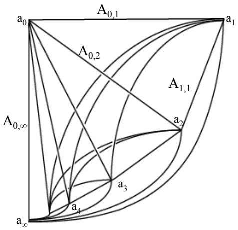


$G \subset [0,1]\times[0,1]$

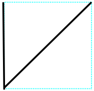
$$A_{i,j} = \{(\underbrace{0,0,\ldots,0}_{i}, \underbrace{x, x,\ldots, x}_{j},1,1,\ldots) \mid 0 \le x \le 1\}$$

$$a_i = (\underbrace{0,0,\ldots,0}_{i},1,1,\ldots)$$

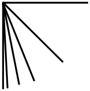
$$\lim_{\leftarrow} G = \bigcup_{i=0}^{\infty}\bigcup_{j=0}^{[\infty]} A_{i,j} \approx \text{ a complete graph with vertices } \{a_i\}$$



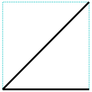
Examples.



$G \subset [0,1] \times [0,1]$



$\varprojlim G$



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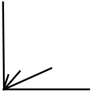

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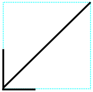

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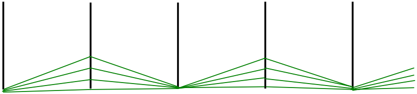
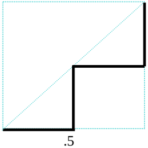


$\dim(\varprojlim G) = \infty$

Example



$G \subset [0,1]\times[0,1]$

$\varprojlim G \approx$ 

Theorem

$[0,1] \times [0,1]$ *is not homeomorphic to* $\varprojlim G$ *for any closed set* $G \subset [0,1] \times [0,1]$.

Proof

Suppose $M = \varprojlim G \approx [0,1] \times [0,1]$.

If $x \in (0,1)$ then $\pi_1^{-1}(x)$ separates M. $\implies dim(\pi_1^{-1}(x)) > 0$.

$\implies \pi_1^{-1}(x)$ contains a nondegenerate continuum L.

Suppose $J = \pi_4(L)$ is the first nondegerate projection of L.

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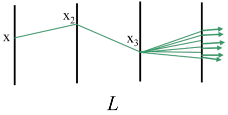

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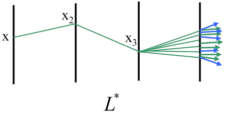

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Then $L^* \approx \sigma(L^*) \approx \sigma^2(L^*) \approx \sigma^3(L^*) = \pi_1^{-1}(J)$.

But $int(\pi_1^{-1}(J)) \neq \emptyset$, $\implies dim(\pi_1^{-1}(J)) = dim(L^*) = 2$.

$\implies dim(\pi_1^{-1}(x)) = 2 \implies int(\pi_1^{-1}(x)) \neq \emptyset$ for each $x \in (0,1)$. $\Rightarrow\Leftarrow$.

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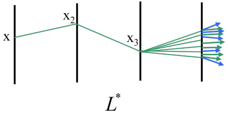

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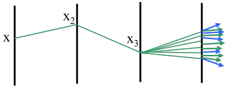

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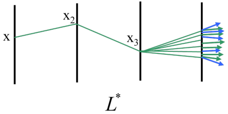

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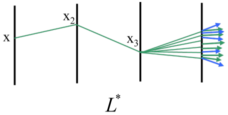

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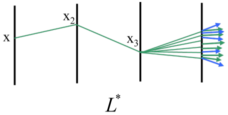

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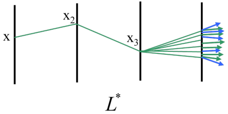

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For what continua X and M is the following statement true?

$**M \approx \varprojlim G$ for some closed subset $G \subset X \times X$ such that $\pi_1(G) = \pi_2(G) = X$.

$**$ is not true if every subcontinuum of X has nonempty interior and M is the countable union of continua K with the following properties for some for $n > 1$.

1. A subset of K has nonempty interior if and only if it has dimension n.
2. K cannot be separated by a zero dimensional set.

More specifically :

$**$ is not true if X is a finite graph and the continuum M is the countable union of n dimensional manifolds for some $n > 1$.

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Some Questions

1. Is there a one dimensional continuum that is not homeomorphic to $\varprojlim G$ for any closed $G \subset [0,1] \times [0,1]$? What about a simple triod (Ingram's question) or a simple closed curve?
2. If X is a continuum, can $X \times [0,1]$ be homeomorphic to $\varprojlim G$ for some closed $G \subset [0,1] \times [0,1]$? (The answer is no if X is a countable union of arcs.) What about $X \times X$?
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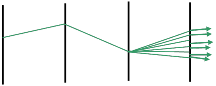
The Dimension of $\varprojlim(f)$

Theorem

If X is a finite graph and $f : X \to 2^X$ is an usc function with $dim(f^{-1}(x)) = 0$ for each $x \in X$, then $dim(\varprojlim f) \leq 1$

Proof

$$dim(f^{-1}(x)) = 0 \ \forall x \in X \implies dim(\pi_1^{-1}(x)) = 0 \ \forall x \in X$$



If K is a nondegenerate subcontinuum of $\varprojlim f$, then $\pi_1(K)$ is a nondegenerate subcontinuum of X.

Since X is a finite graph $\pi_1(K)$ is separated by a finite set $\{x_1, x_2, \ldots, x_m\}$.

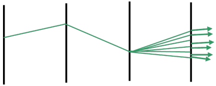
The zero dimensional set $\bigcup \pi_1^{-1}(x_i)$ separates K .

Therefore every subcontinuum of $\varprojlim f$ is separated by a zero dimensional set.

$\implies dim(\varprojlim f) \leq 1.$

Proof

$$dim(f^{-1}(x)) = 0\ \forall x \in X \implies dim(\pi_1^{-1}(x)) = 0\ \forall x \in X$$



If K is a nondegenerate subcontinuum of $\underleftarrow{\lim} f$, then $\pi_1(K)$ is a nondegenerate subcontinuum of X.

Since X is a finite graph $\pi_1(K)$ is separated by a finite set $\{x_1, x_2, \ldots, x_m\}$.

The zero dimensional set $\bigcup \pi_1^{-1}(x_i)$ separates K .

Therefore every subcontinuum of $\underleftarrow{\lim} f$ is separated by a zero dimensional set.

$\implies dim(\underleftarrow{\lim} f) \leq 1.$

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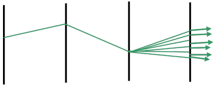

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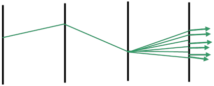

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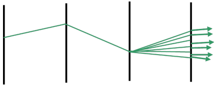

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Theorem

If X is compact and $f : X \to 2^X$ is an usc function with $dim(f(x)) = 0$ for each $x \in X$, then $dim(\varprojlim f) \leq dim(X)$.

Observations

- If $G \subset [0,1] \times [0,1]$ then $dim(\varprojlim G) > 1$ only if G contains a vertical line and a horizontal line. Examples

- If $f : X \to X$ and $g : X \to X$ are continuous and $F(x) = \{f(x), g(x)\}$, then $dim(\varprojlim F) \leq dim(X)$.

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