FINITE GRAPHS THAT ARE INVERSE LIMITS WITH A SET VALUED FUNCTION ON [0,1]

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Abstract. We show that a finite graph that is the inverse limit with a single surjective upper semi-continuous set valued function $f : [0,1] \to 2^{[0,1]}$ is either an arc or a simple triod. It is not known if there is such a simple triod.

1. Introduction

Let M be a closed subset of $[0,1]\times[0,1]$, then $\varprojlim M$, the generalized inverse limit of M, is defined as $\varprojlim M = \{(x_1, x_2, \ldots) \in \Pi_{i=1}^{\infty}[0,1] \mid (x_i, x_{i-1}) \in M \text{ for } i \in \{2,3,\ldots\}\}$. If M is the graph of a continuous function $f : [0,1] \to [0,1]$ then this is a traditional inverse limit and the result is an arc-like or chainable continuum. Traditional inverse limits have been used to study the structure of continua, to construct continua with complicated structures, and to study the dynamics of the bonding functions. In recent years a number of papers have been written about generalized inverse limits starting with Mahavier in [5]. In economic modeling and other areas the dynamics of set valued functions defined on an interval are important. Understanding generalized inverse limits can contribute to the study of dynamics in part because the space of forward orbits of a set valued function is the same as the inverse limit of the inverse of the function. We are just beginning to understand generalized inverse limits and it is not yet clear what sorts of continua can be obtained in this way. The generalized inverse limit with a closed subset of $[0,1] \times [0,1]$ will be compact but not necessarily connected, even for a connected set M. Examples have been given in [5][2] [3][7] and several other papers, of generalized inverse limits with closed subsets of $[0,1] \times [0,1]$ that are homeomorphic to various dendrites, a harmonic fan, infinite dimensional continua, and even a Cantor set [4]. Of course an arc is the inverse limit with M equal to the diagonal in $[0,1] \times [0,1]$, but other than that no example has been found of a closed subset of $[0,1] \times [0,1]$ whose generalized inverse limit is a finite graph. There are a few continua that are known not to be obtainable as such a generalized inverse limit, for example any n-cell with $n > 1$ [6], and a simple closed curve [1]. Until now the simple closed curve was

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the only one dimensional example. We will show that no finite graph except an arc and possibly a simple triod can be the inverse limit with a single surjective upper semi-continuous bonding function from $[0,1]$ to the nonempty closed subsets of $[0,1]$.

2. Notation and Preliminary Results

A continuum is a compact connected metric space. A subcontinuum A of a continuum X is a free arc if A is an arc such that the boundary of A in X is contained in the set of endpoints of A, and a continuum X is a finite graph if X is the union of a finite number of free arcs. A simple triod is an acyclic finite graph with three endpoints. The Hilbert cube is the product $\Pi_{i=1}^{\infty}[0,1]$ with metric given by $d(x,y) = \sum_{i=1}^{\infty} \frac{|\pi_i(x)-\pi_i(y)|}{2^i}$ where $\pi_i : \Pi_{i=1}^{\infty}[0,1] \to [0,1]$ is defined by $\pi_i(x) = \pi_i((x_1, x_2, x_3, \ldots)) = x_i$. The function $\pi_{1,n} : \Pi_{i=1}^{\infty}[0,1] \to \Pi_{i=1}^{n}[0,1]$ is defined by $\pi_{1,n}(x) = \pi_{1,n}((x_1, x_2, x_3, \ldots)) = (x_1, \ldots x_n)$. The binary operation $\oplus : \Pi_{i=1}^{n}[0,1] \times \Pi_{i=1}^{\infty}[0,1] \to \Pi_{i=1}^{\infty}[0,1]$ is defined by $(x_1, \ldots, x_n) \oplus (y_1, y_2, \ldots) = (x_1, \ldots, x_n, y_1, y_2, \ldots)$. The continuous function $\sigma : \Pi_{i=1}^{\infty}[0,1] \to \Pi_{i=1}^{\infty}[0,1]$ is defined by $\sigma((x_1, x_2, x_3, \ldots)) = (x_2, x_3, \ldots)$.

A set valued function $f : X \to 2^Y$ into the nonempty closed subsets of Y is upper semi-continuous (usc) if for each open set $V \subset Y$ the set $\{x : f(x) \subset V\}$ is an open set in X. A set valued function $f : X \to 2^Y$ is usc if and only if the graph of f, $\Gamma(f) = \{(x,y) \mid y \in f(x)\}$, is a closed subset of $X \times Y$ [3, Theorem 2.1 p. 120]. A set valued function $f : X \to 2^Y$ will be called surjective if for each $y \in Y$ there is a point $x \in X$ such that $y \in f(x)$. The inverse limit of the sequence of set valued functions $\{f_i\}$, where $f_i : X_{i+1} \to 2^{X_i}$ is denoted $\varprojlim f_i$, and is defined to be the set of all $(x_1, x_2, x_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$ such that $x_i \in f_i(x_{i+1})$ for each i. The functions f_i are called bonding maps. The study of inverse limits with upper semi-continuous set valued functions began with Mahavier in [5] and Mahavier and Ingram in [3].

In this paper we are considering inverse limits with a single surjective upper semi-continuous bonding function $f : [0,1] \to 2^{[0,1]}$ which will be denoted $\varprojlim f$. It is easy to see that $\sigma(\varprojlim f) = \varprojlim f$. The results in this paper are based on an examination of the orbits of open sets in $\varprojlim f$ under the shift map σ. A technique used by Illanes in [1] to show that a simple closed curve is not the inverse limit of a single surjective set valued bonding map on $[0,1]$ will be used repeatedly. It involves the observation that if $x, y \in \varprojlim f$ such that $\pi_n(x) = \pi_1(y)$, then $\pi_{1,n-1}(x) \oplus y \in \varprojlim f$, and if n is large then $\pi_{1,n-1}(x) \oplus y$ is a point that is close to x and $\sigma^{n-1}(\pi_{1,n-1}(x) \oplus y) = y$. We will exploit this simple idea using uniformizations to determine the orbits of open subsets under the shift map.

Note that if a simple closed curve is embedded in $[0,1] \times [0,1]$ so that it is the graph of a surjective set valued function f_1, and f_i is the identity function for

$i > 1$, then $\varprojlim f_i$ is a simple closed curve. Since something like this can be done for any continuum embeddable in $[0,1]^n$, restricting the question to what continua are obtainable as an inverse limit with a single set valued bonding function is necessary.

We will also restrict our attention to surjective bonding maps. Its easy to see that if $f : [0,1] \to 2^{[0,1]}$ is not necessarily surjective and $\varprojlim f$ is a continuum then $\pi_i(\varprojlim f)$ is an interval for each i. Moreover if $(x_1, x_2, \dots) \in \varprojlim f$, then $(x_2, x_3, \dots) \in \varprojlim f$. Also if $(x_1, x_2, \dots) \in \varprojlim f$, then there is a $y \in f(x_1)$ so that $(y, x_1, x_2, \dots) \in \varprojlim f$ and $\sigma((y, x_1, x_2, \dots)) = ((x_1, x_2, \dots))$. Therefore $\sigma(\varprojlim f) = \varprojlim f$. Thus $\pi_i(\varprojlim f) = \pi_1(\varprojlim f) = [a, b]$ for some $a, b \in [0,1]$. For convenience we will assume $a = 0$ and $b = 1$. Note also that we do assume that $f(x)$ is not empty for each $x \in [0,1]$. Consider the set $M = \{(x, y) \in [0, \frac{1}{2}] \times [0,1] \,|\, y \in \{x, 1-x, \frac{3}{4} - \frac{1}{2}x\}\}$. It is easy to see that $(x_1, x_2, x_3, \dots) \in \varprojlim f$ if and only if $(x_2, x_1) \in M$ and $x_j = x_2$ for each $j \in \{3, 4, 5, \dots\}$. Thus the simple triod M is homeomorphic to $\varprojlim M$. Essentially the first bonding function is not the same as the rest of the bonding functions, and we can eliminate this sort of example with the assumption that $f(x)$ is a nonempty closed subset of $[0,1]$ for each $x \in [0,1]$.

The functions α and β in the following lemma form what we call a discrete uniformization of the functions f and g. For an elementary and elegant proof of the following lemma see [1]. For each $n \in \mathbb{N}$, let $S_n = \{0, \dots, n\}$.

Lemma 1. *Let $f, g : [0,1] \to [0,1]$ be continuous functions such that $f(0) = 0 = g(0)$ and $f(1) = 1 = g(1)$, and let $\delta > 0$. Then there exist $n \in \mathbb{N}$ and functions $\alpha, \beta : S_n \to [0,1]$ such that $f \circ \alpha = g \circ \beta$, $\alpha(0) = 0 = \beta(0)$, $\alpha(n) = 1 = \beta(n)$ and, for each $i \in \{0, \dots, n-1\}$, $|\alpha(i+1) - \alpha(i)| < \delta$ and $|\beta(i+1) - \beta(i)| < \delta$.*

We will also use a variation of Lemma 1 also proved in [1] .

Lemma 2. *Let $f, g : [0,1] \to [0,1]$ be continuous functions such that $f(0) = 0$ and $f(1) = 1$, and let $\delta > 0$. Then there exist $n \in \mathbb{N}$ and functions $\alpha, \beta : S_n \to [0,1]$ such that $f \circ \alpha = g \circ \beta$, $\beta(0) < \delta$, $1 - \beta(n) < \delta$ and, for each $i \in \{0, \dots, n-1\}$, $|\alpha(i+1) - \alpha(i)| < \delta$ and $|\beta(i+1) - \beta(i)| < \delta$.*

The remaining results in this section are technical facts about finite graphs that come up in the proofs of the main results in the next section.

Lemma 3. *If G is a finite graph with metric d, and $g : [0,1] \to B$ is a homeomorphism onto an arc B in G, then there is a $\delta > 0$ such that if C is a continuum in G with $d(C, g(0)) < \delta$, $d(C, g(1)) < \delta$, and $d(c, B) < \delta$ for each $c \in C$, then $g([\frac{1}{4}, \frac{3}{4}]) \subset C$.*

Proof. Suppose G is a finite graph with metric d, and $g : [0,1] \to B$ is a homeomorphism onto an arc B in G. Since G is a finite graph, there is an open set U in

G containing the arc B such that U is arc connected and acyclic. There is a $\delta_1 > 0$ such that if $x \in G$ and $d(x, B) < \delta_1$ then $x \in U$. Since $U \setminus g([\frac{1}{4}, \frac{3}{4}]$ has finitely many components, the component of $U \setminus g([\frac{1}{4}, \frac{3}{4}])$ that contains $g(0)$ is open in G. Therefore there is a $\delta_2 > 0$ such that if $x \in G$ and $d(g(o), x) < \delta_2$ then x is in the component of $U \setminus g([\frac{1}{4}, \frac{3}{4}])$ that contains $g(0)$. Similarly there is a $\delta_3 > 0$ such that if $x \in G$ and $d(g(1), x) < \delta_3$ then x is in the component of $U \setminus g([\frac{1}{4}, \frac{3}{4}])$ that contains $g(1)$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Suppose C is an continuum in G with $d(C, g(0)) < \delta$, $d(C, g(1)) < \delta$, and $d(c, B) < \delta$ for each $c \in C$. Then $C \subset U$, and since the components of $U \setminus g([\frac{1}{4}, \frac{3}{4}])$ are arc connected there is an arc $D_1 \subset U \setminus g([\frac{1}{4}, \frac{3}{4}])$ such that $g(0) \in D_1$ and $C \cap D_1 \neq \emptyset$ and there is an arc $D_2 \subset U \setminus g([\frac{1}{4}, \frac{3}{4}])$ such that $g(1) \in D_2$ and $C \cap D_2 \neq \emptyset$. Since U is acyclic $g([\frac{1}{4}, \frac{3}{4}]) \subset C \cup D_1 \cup D_2$. So $g([\frac{1}{4}, \frac{3}{4}]) \subset C$. □

Lemma 4. *Suppose G is a finite graph with metric d, E and F are connected closed subsets of G and $\{\epsilon_i\}$ is a sequence of positive numbers converging to 0 such that for each $i \in \mathbb{N}$ there is an integer $m_i \in \mathbb{N}$ and a function $\Psi_i : S_{m_i} \to G$ such that for each $j \in \{0, \dots, m_i - 1\}$, $d(\Psi_i(j+1), \Psi_i(j)) < \epsilon_i$ and there exist $j, k \in S_{m_i}$ such that $d(\Psi_i(j), E) < \epsilon_i$, and $d(\Psi_i(k), F) < \epsilon_i$. Then the closure of $\bigcup_{i=1}^{\infty} \bigcup_{j=0}^{m_i} \Psi_i(j)$ contains an arc with one endpoint in E and the other endpoint in F.*

Proof. Let W be the closure of $\bigcup_{i=1}^{\infty} \bigcup_{j=0}^{m_i} \Psi_i(j)$. Assume that if D is an arc in G with one endpoint in E and the other endpoint in F then D is not contained in W. There are only finitely many arcs $\{D_1, \dots, D_k\}$ in G that are irreducible between E and F. For each $j \in \{1, \dots, k\}$ there is open set $U_j \subset D_j \setminus W$ such that U_j is homeomorphic to $(0, 1)$ and G has order two at each point in U_j. Since G is a finite graph, $\bigcup_{j=1}^{k} U_j$ separates G into a finite number of components with E and F in different components. Let $\delta > 0$ be less than the smallest distance between any two components of $G \setminus \bigcup_{j=1}^{k} U_j$. Choose $i \in \mathbb{N}$ such that $\epsilon_i < \frac{\delta}{2}$. Then $\bigcup_{j=0}^{m_i} \Psi_i(j)$ is contained in a single component of $G \setminus \bigcup_{j=1}^{k} U_j$ and that component must contain an arc which is irreducible between E and F, which is a contradiction. So W contains an arc with one endpoint in E and the other endpoint in F. □

3. Main Results

Throughout this section assume $f : [0, 1] \to 2^{[0,1]}$ is surjective upper semi-continuous set valued function and $G = \varprojlim f$ is a non degenerate finite graph. Let P_0 be a point in G such that $\pi_1(P_0) = 0$, and let P_1 be a point in G such that $\pi_1(P_1) = 1$. Let A be an arc in G with endpoints P_0 and P_1. A Type I point in G is a point x such that there exists an $N \in \mathbb{N}$ such that $\sigma^n(x) \in A$ for each $n \geq N$. A type II point in G is a point that is not Type I. A point x in G has a Type I neighborhood if x is contained in an open set O such that there exists an $N \in \mathbb{N}$ such that $\sigma^n(O) \subset A$ for each $n \geq N$.

Lemma 5. *Every point of G that is not an endpoint of G has a Type I neighborhood.*

Proof. Let x be a point of G that is not an endpoint of G. There is an arc B in G and a homeomorphism $g : [0, 1] \to B$ onto the arc B such that $g(\frac{1}{2}) = x$ and G has order two at each point of $B \setminus \{g(\frac{1}{2})\}$. According to Lemma 3 there is a $\delta_1 > 0$ such that if C is an arc in G with $d(C, g(0)) < \delta_1$, $d(C, g(1)) < \delta_1$, and $d(c, B) < \delta_1$ for each $c \in C$, then $g([\frac{1}{4}, \frac{3}{4}]) \subset C$. Let E and F be closed arcs in G with diameter less than δ_1 such that $g(0)$ is contained in the interior of E, and $g(1)$ is contained in the interior of F, and such that $(E \cup F) \cap g([\frac{1}{4}, \frac{3}{4}]) = \emptyset$. There is a $\delta_2 > 0$ such that if $x \in G$ and $d(x, g(0)) < \delta_2$ then $x \in E$, and if $y \in G$ and $d(y, g(1)) < \delta_2$ then $y \in F$. Let $\delta = \min\{\delta_1, \delta_2\}$.

Let $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{\delta}{2}$. Let $f : [0, 1] \to A$ be a homeomorphism such that $f(0) = P_0$ and $f(1) = P_1$. For $n \geq N$, $\pi_n \circ g : [0, 1] \to [0, 1]$ and $\pi_1 \circ f : [0, 1] \to [0, 1]$ are continuous functions and $\pi_1 \circ f(0) = 0$ and $\pi_1 \circ f(1) = 1$. Let $\epsilon > 0$ such that $\epsilon < \delta$. Since f and g are uniformly continuous there exists $\epsilon' > 0$ such that if $a, b \in [0, 1]$ and $|b - a| < \epsilon'$ then $d(f(a), f(b)) < \frac{\epsilon}{2}$ and $d(g(a), g(b)) < \frac{\epsilon}{2}$. According to Lemma 2 there is an $m \in \mathbb{N}$ and functions $\alpha, \beta : S_m \to [0, 1]$ such that $\pi_1 \circ f \circ \alpha = \pi_n \circ g \circ \beta$, $\beta(0) < \epsilon'$, $1 - \beta(m) < \epsilon'$, and $|\alpha(i + 1) - \alpha(i)| < \epsilon'$ and $|\beta(i + 1) - \beta(i)| < \epsilon'$ for each $i \in \{0, \ldots, m - 1\}$.

Now define $\Psi : S_m \to G$ by $\Psi(i) = \pi_{1,n-1}(g \circ \beta(i)) \oplus f \circ \alpha(i)$ for each $i \in \{0, \ldots, m\}$. Then for each $i \in \{0, \ldots, m\}$, $\Psi(i) \in G$, $\sigma^{n-1}(\Psi(i)) = f \circ \alpha(i) \in A$, and $d(\Psi(i), B) \leq d(\Psi(i), g \circ \beta(i)) < \frac{1}{2^N} < \delta$. For each $i \in \{0, \ldots, m - 1\}$, $d(\Psi(i + 1), \Psi(i)) < d(g \circ \alpha(i + 1), g \circ \alpha(i)) + d(f \circ \beta(i + 1), f \circ \beta(i)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$. Also $d(\Psi(0), g(0)) < d(\Psi(0), g \circ \beta(0)) + d(g \circ \beta(0), g(0)) < \frac{1}{2^n} + \frac{\epsilon}{2} < \frac{\delta}{2} + \frac{\epsilon}{2} < \delta$, and similarly $d(\Psi(m), g(1)) < \delta$. Therefore $\Psi(0) \in E$ and $\Psi(m) \in F$.

Let $\{\epsilon_i\}$ be a sequence of positive numbers less than δ and converging to 0. Construct $\Psi_i : S_{m_i} \to G$ as above for each ϵ_i, leaving n and δ fixed, and let W be the closure of $\bigcup_{i=1}^{\infty} \bigcup_{j=0}^{m_i} \Psi_i(j)$. By Lemma 4, W contains an arc W' such that $W' \cap E \neq \emptyset$ and $W' \cap F \neq \emptyset$. Let $C = W' \cup E \cup F$. For each i and each j in the domain of Ψ_i, $d(\Psi_i(j), B) < \delta \leq \delta_1$. Therefore for each $x \in C$, $d(x, B) < \delta_1$. It follows from Lemma 3 and the choice of δ_1 that $g([\frac{1}{4}, \frac{3}{4}]) \subset C$. Since $(E \cup F) \cap g([\frac{1}{4}, \frac{3}{4}]) = \emptyset$, $g([\frac{1}{4}, \frac{3}{4}]) \subset W' \subset W$. Now since for each i and each j in the domain of Ψ_i, $\sigma^{n-1}(\Psi_i(j)) \in A$ and since σ^{n-1} is continuous and A is closed, $\sigma^{n-1}(W) \subset A$. Therefore $\sigma^{n-1}(g([\frac{1}{4}, \frac{3}{4}])) \subset A$. The choice of $n \geq N$ was arbitrary so $\sigma^n(g([\frac{1}{4}, \frac{3}{4}])) \subset A$ for each $n \geq N - 1$.

For each $x \in G$ that is not an endpoint of G there is a finite collection of arcs $\{B_1, \ldots, B_k\}$ in G such that for each $i \in \{1, \ldots, k\}$ there is a homeomorphism

$g_i : [0,1] \to B_i$ onto the arc B_i such that $g_i(\frac{1}{2}) = x$ and G has order two at each point of $B_i \setminus g_i(\frac{1}{2})$, and x is contained in the interior of $\bigcup_{i=1}^{k} g_i([\frac{1}{4}, \frac{3}{4}])$. There is an $N \in \mathbb{N}$ such that if $n \geq N$, then $\sigma^n(\bigcup_{i=1}^{k} g_i([\frac{1}{4}, \frac{3}{4}])) \subset A$. Therefore $\bigcup_{i=1}^{k} g_i([\frac{1}{4}, \frac{3}{4}])$ is a Type I neighborhood of x. □

Theorem 6. *If $f : [0,1] \to 2^{[0,1]}$ is a surjective upper semi-continuous function such that $\varprojlim f$ is a finite graph, then $\varprojlim f$ must have at least one endpoint.*

Proof. Suppose G is a non degenerate finite graph with no endpoints and G is the inverse limit of a surjective upper semi-continuous function $f : [0,1] \to 2^{[0,1]}$. According to Lemma 5 there is a cover of G consisting of Type I neighborhoods. Since G is compact there is an $n \in \mathbb{N}$ such that $\sigma^n(G)$ is contained in an arc. Since $\sigma^n(G) = G$, this is a contradiction. □

Lemma 7. *G has a finite number of Type II points, and for each Type II point x the following are true:*

i. *There is an $n \in \mathbb{N}$ such that $\sigma^n(x) = x$.*
ii. *If $i \in \mathbb{N}$ and $y \in G$ such that $\pi_i(x) = \pi_i(y)$ then $\pi_{1,i}(x) = \pi_{1,i}(y)$.*
iii. *If $n \in \mathbb{N}$ then for each $\epsilon > 0$ there is an $\delta > 0$ such that if $y \in G$ and $|\pi_n(x) - \pi_n(y)| < \delta$ then $\sum_{i=1}^{n} \frac{|\pi_i(x) - \pi_i(y)|}{2^i} < \epsilon$.*

Proof. From Lemma 5 it follows that every non endpoint in G is a Type I point. Therefore, since G has a finite number of endpoints, G has a finite number of Type II points. It is easy to see that if $y \in G$ is Type I, then $\sigma^m(y)$ is Type I for each $m \in \mathbb{N}$, and if $x \in G$ is Type II then $\sigma^m(x)$ is Type II for each $m \in \mathbb{N}$. Therefore σ maps the Type II points onto the Type II points, and thus σ permutes the Type II points. So if $x \in G$ is Type II, then there is an $n \in \mathbb{N}$ such that $\sigma^n(x) = x$.

Suppose x is a Type II point and $y \in G$ such that for some $i \in \mathbb{N}$, $\pi_i(x) = \pi_i(y)$ and suppose $\pi_{1,i}(x) \neq \pi_{1,i}(y)$. Then G contains the point $z = \pi_{1,i}(y) \oplus \sigma^i(x)$. But for each n, $\sigma^n(z) \neq z$. Therefore z is Type I even though there is an m such that $\sigma^m(z) = x$. This is a contradiction so $\pi_{1,i}(x) = \pi_{1,i}(y)$.

Finally, let $x \in G$ be Type II, let $n \in \mathbb{N}$, and let $\epsilon > 0$. Define $F_n : [0,1] \to 2^{[0,1]^n}$ by $F_n(s) = \{(t_1, \ldots, t_n) \,|\, t_n = s \text{ and } t_i \in f(t_{i+1}) \text{ for } i \in \{1, \ldots, n-1\}\}$. According to Lemma 5.1 of [6], F_n is upper semi-continuous. Since $f(\pi_{i+1}(x)) = \{\pi_i(x)\}$ for each $i \in \mathbb{N}$, $F_n(\pi_n(x)) = \{\pi_{1,n}(x)\}$. Let $U = \prod_{i=1}^{n} (\pi_i(x) - \epsilon, \pi_i(x) + \epsilon)$. Then U is an open set in $[0,1]^n$ and $F_n(\pi_n(x)) \subset U$. So there is a $\delta > 0$ such that if $y \in G$ and $|\pi_n(y) - \pi_n(x)| < \delta$ then $F_n(\pi_n(y)) \subset U$. Since $\pi_{1,n}(y) \in F_n(\pi_n(y))$ it follows that $\sum_{i=1}^{n} \frac{|\pi_i(x) - \pi_i(y)|}{2^i} < \epsilon \sum_{i=1}^{n} \frac{1}{2^i} < \epsilon$. □

Part iii of Lemma 7 will play an important role in the results that follow. Part iii implies that if $\{z_n\}$ is a sequence of points in this finite graph $G = \varprojlim f$, and there is a Type II point $z \in G$, and an $m \in \mathbb{N}$ such that $\sigma^m(z) = z$, and $\{\sigma^m(z_n)\}$ converges to z, then $\{z_n\}$ also converges to z. This will be used to assure that the closure of $\bigcup_{i=1}^{\infty} \bigcup_{j=0}^{m_i} \Psi_i(j)$ contains a Type II point, where Ψ_i is constructed as in the proof of Lemma 5.

Lemma 8. *G has at most two Type II points.*

Proof. Suppose e_1, e_2, and e_3 are three distinct Type II points, and let $n \in \mathbb{N}$ such that $\sigma^n(e_i) = e_i$ for each $i \in \{1, 2, 3\}$. Let $a = \pi_1(e_1)$, $b = \pi_1(e_2)$, and $c = \pi_1(e_3)$. If $a = b$ then $e_1 = e_2$ since for each $k \in \mathbb{N}$, $\pi_{nk}(e_1) = \pi_{nk}(e_2)$ and therefore $\pi_{1,nk}(e_1) = \pi_{1,nk}(e_2)$ by Lemma 7. So $a \neq b$ and similarly $b \neq c$ and $c \neq a$. Without loss of generality assume $a < b < c$. Suppose K is a continuum in G that contains e_1 and e_3. Then for each k, K contains a point z_k such that $\pi_{nk}(z_k) = b = \pi_{nk}(e_2)$. Therefore by Lemma 7, $\pi_{1,nk}(z_k) = \pi_{1,nk}(e_2)$. It follows that the sequence $\{z_k\}$ converges to e_2 and therefore $e_2 \in K$. But e_2 is Type II and thus e_2 is an endpoint of G by Lemma 5. So it is not possible that every continuum that contains e_1 and e_3 contains e_2. □

For the remainder of this section we will consider the cases where G has zero, one, or two Type II points and show that in each case either G must be an arc, or G must be either an arc or a simple triod. We begin with the case where G has no Type II points for which the next lemma is useful.

Lemma 9. *Every Type I point in G has a Type I neighborhood.*

Proof. Every non endpoint of G has a Type I neighborhood, and if $x \in G$ and there is an m such that $\sigma^m(x)$ has a Type I neighborhood, then by the continuity of σ, x has a Type I neighborhood. So if G contains a Type I point x such that x does not have a Type I neighborhood, then for each $m \in \mathbb{N}$, $\sigma^m(x)$ is an endpoint of G. Since x is Type I there is an $N \in \mathbb{N}$ such that for each $m \geq N$, $\sigma^m(x) \in A$, it follows that for each $m \geq N$, $\sigma^m(x) \in \{P_0, P_1\}$. It follows that there is no Type I point that does not have a Type I neighborhood if each of the following is true:
(i) If $\sigma(P_0) = P_0$ and P_0 is an endpoint of G then P_0 has a Type I neighborhood.
(ii) If $\sigma(P_1) = P_1$ and P_1 is an endpoint of G then P_1 has a Type I neighborhood.
(iii) If $\sigma(P_0) = P_1$, $\sigma(P_1) = P_0$, P_0 and P_1 are both endpoints of G then P_0 and P_1 have Type I neighborhoods.

Assume $\sigma(P_0) = P_0$ and P_0 is an endpoint of G. It follows that for each $i \in \mathbb{N}$, $\pi_i(P_0) = 0$, and there is an arc L in G such that P_0 is contained in the interior of L, and G has order two each point in $L \setminus P_0$, and $P_1 \in G \setminus L$. Then there is a $\delta_1 > 0$ such that if $x \in G$ and $d(x, P_1) < \delta$ then $x \in G \setminus L$. Let F be an closed connected subset of G with diameter less than δ_1 such that P_1 is contained in the interior of F. There is a $\delta_2 > 0$ such that if $x \in G$ and $d(x, P_1) < \delta_2$ then $x \in F$. Let $\delta = \min\{\delta_1, \delta_2\}$.

Let $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{\delta}{2}$. Let $n \geq N$ and let $\epsilon > 0$. Let R_1 be a point in G such that $\pi_n(R_1) = 1$. And let $g : [0,1] \to G$ be a continuous function such that $g(0) = P_0$ and $g(1) = R_1$, and $g(\frac{1}{2}) = P_1$.

Let $f : [0,1] \to A$ be a continuous function such that $f(0) = P_0$ and $f(1) = P_1$. Then $\pi_1 \circ f : [0,1] \to [0,1]$ and $\pi_n \circ g : [0,1] \to [0,1]$ are continuous functions with $\pi_1 \circ f(0) = 0 = \pi_n \circ g(0)$ and $\pi_1 \circ f(1) = 1 = \pi_n \circ g(1)$. There is an $\epsilon_1 > 0$ such that if $s, t \in [0,1]$ and $|s - t| < \epsilon_1$ then $d(f(s), f(t)) < \min\left\{\frac{\epsilon}{2}, \frac{\delta}{2}\right\}$ and $d(g(s), g(t)) < \min\left\{\frac{\epsilon}{2}, \frac{\delta}{2}\right\}$. According to Lemma 1 there is an $m \in \mathbb{N}$ and functions $\alpha, \beta : S_m \to [0,1]$ such that $\pi_1 \circ f \circ \alpha = \pi_N \circ g \circ \beta$, $\alpha(0) = 0 = \beta(0)$, $\alpha(1) = 1 = \beta(1)$, and for each $i \in \{0, \ldots, m-1\}$, $|\alpha(i+1) - \alpha(i)| < \epsilon_1$ and $|\beta(i+1) - \beta(i)| < \epsilon_1$.

Define $\Psi : S_m \to G$ by $\Psi(i) = \pi_{1,n-1}(g \circ \beta(i)) \oplus f \circ \alpha(i)$ for each $i \in \{0, \ldots, m\}$. Then for each $i \in \{0, \ldots, m\}$, $\Psi(i) \in G$ and $\Psi(0) = P_0$. Also for each $i \in \{0, \ldots, m-1\}$, $d(\Psi(i+1), \Psi(i)) < d(g \circ \alpha(i+1), g \circ \alpha(i)) + d(f \circ \beta(i+1), f \circ \beta(i)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$. There is an $i_0 \in P_m$ such that $\frac{1}{2}$ is between $\beta(i_0)$ and $\beta(i_0 + 1)$. Then $d(\Psi(i_0), P_1) = d(\Psi(i_0), g(\frac{1}{2})) \leq d(g \circ \beta(i_0), g(\frac{1}{2})) + \frac{1}{2^N} < \frac{\delta}{2} + \frac{\delta}{2} = \delta$. Thus $\Psi(i_0) \in F$. Finally note that $\sigma^{n-1}(\Psi(i)) = f \circ \alpha(i) \in A$ for each $i \in S_m$.

Let $\{\epsilon_i\}$ be a sequence of positive numbers less than δ and converging to 0. Construct $\Psi_i : S_{m_i} \to G$ as above for each ϵ_i, leaving n and δ fixed, and let W be the closure of $\bigcup_{i=1}^{\infty} \bigcup_{j=0}^{m_i} \Psi_i(j)$. By Lemma 4, W contains an arc W' with one endpoint in $E = \{P_0\}$ and the other endpoint in F. Therefore $L \subset W'$. Now since for each i and each j in the domain of Ψ_i, $\sigma^{n-1}(\Psi_i(j)) \in A$, and since σ^{n-1} is continuous and A is closed, $\sigma^{n-1}(W) \subset A$. Therefore $\sigma^{n-1}(L) \subset A$. The choice of $n \geq N$ was arbitrary so $\sigma^n(L) \subset A$ for each $n \geq N - 1$. Therefore L is a Type I neighborhood of P_0.

The proof that if $\sigma(P_1) = P_1$ and P_1 is an endpoint of G then P_1 has a Type I neighborhood is the same as the proof above. So assume $\sigma(P_0) = P_1$ and $\sigma(P_1) = P_0$ and both P_0 and P_1 are endpoints of G . In this case the proof above with each instance of "n" replaced with "$2n$" shows that there is an $N \in \mathbb{N}$ and open sets O_0 and O_1 such that $P_0 \in O_0$ and $P_1 \in O_1$ and such that for each $n \geq N$, $\sigma^{2n}(O_0) \subset A$ and $\sigma^{2n}(O_1) \subset A$. Assume $x \in \sigma^{-1}(O_1) \cap O_0$ and $n \geq 2N$. Then if n is even, since $x \in O_0$, $\sigma^n(x) \in A$. If n is odd, there is an $m \geq N$ such that $n - 1 = 2m$, and since $\sigma(x) \in O_1$, $\sigma^n(x) = \sigma^{n-1}(\sigma(x)) = \sigma^{2m}(\sigma(x)) \in A$. Therefore $\sigma^{-1}(O_1) \cap O_0$ is a Type I neighborhood of P_0 and similarly $\sigma^{-1}(O_0) \cap O_1$ is a Type I neighborhood of P_1. □

Theorem 10. *If G has no Type II points then G is an arc.*

Proof. Since G is compact and every point of G is contained in a Type I neighborhood, G can be covered by a finite collection of Type I neighborhoods. Therefore there is an $N \in \mathbb{N}$ such that if $n > N$, $\sigma^N(G) = G \subset A$. $\square$

Theorem 11. *If G has one Type II point, then G is an arc or a simple triod.*

Proof. Let q be the only Type II point of G. Let T be a continuum in G that contains A and is irreducible about the set $\{P_0, P_1, q\}$. Therefore T is either an arc or a simple triod. Since $\sigma(q) = q$, q is not contained in A, and there is a $b \in [0,1]$ such that $\pi_i(q) = b$ for each $i \in \mathbb{N}$. Let $f : [0,1] \to T$ be a continuous function such that $f(0) = P_0$ and $f(\frac{1}{2}) = q$, and $f(1) = P_1$. Let $\delta > 0$ such that $2\delta < d(q, A)$. Let L be an arc with diameter less than $\frac{\delta}{2}$ that contains q in its interior and such that every point in $L \setminus Q$ has order two. Let F be a connected closed subset of $G \setminus L$ such that if $y \in G$ and $d(y, P_0) < \frac{\delta}{2}$ then $y \in F$. Therefore $F \cap L = \emptyset$.

Since each point of $\overline{G \setminus L}$ has a Type I neighborhood and $\overline{G \setminus L}$ is compact, there is an $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $\sigma^{n-1}(\overline{G \setminus L}) \subset A$. Let R_0 and R_1 be points in G such that $\pi_N(R_0) = 0$ and $\pi_N(R_1) = 1$. And let $g : [0,1] \to G$ be a continuous function such that $g(0) = R_0$, $g(1) = R_1$, and $g(\frac{1}{2}) = P_0$.

Then $\pi_1 \circ f : [0,1] \to [0,1]$ and $\pi_N \circ g : [0,1] \to [0,1]$ are continuous functions with $\pi_1 \circ f(0) = 0 = \pi_N \circ g(0)$ and $\pi_1 \circ f(1) = 1 = \pi_N \circ g(1)$. Let $\epsilon > 0$ such that $\epsilon < \frac{\delta}{4}$. By Lemma 7 there is an $\epsilon_1 > 0$ such that if $q \in Q$ and $y \in G$ such that $|\pi_N(q) - \pi_N(y)| < \epsilon_1$ then $\sum_{i=1}^{N} \frac{|\pi_i(q) - \pi_i(y)|}{2^i} < \frac{\epsilon}{2}$. Also, there is an $\epsilon_2 > 0$ such that if $s, t \in [0,1]$ and $|s - t| < \epsilon_2$ then $d(f(s), f(t)) < \min\left\{\frac{\epsilon}{2}, \epsilon_1\right\}$ and $d(g(s), g(t)) < \min\left\{\frac{\epsilon}{2}, \epsilon_1\right\}$. According to Lemma 1 there is an $m \in \mathbb{N}$ and functions $\alpha, \beta : S_m \to [0,1]$ such that $\pi_1 \circ f \circ \alpha = \pi_N \circ g \circ \beta$, $\alpha(0) = 0 = \beta(0)$, $\alpha(1) = 1 = \beta(1)$, and for each $i \in \{0, \ldots, m-1\}$, $|\alpha(i+1) - \alpha(i)| < \epsilon_2$ and $|\beta(i+1) - \beta(i)| < \epsilon_2$.

Define $\Psi : S_m \to G$ by $\Psi(i) = \pi_{1,N-1}(g \circ \beta(i)) \oplus f \circ \alpha(i)$. Then for each $i \in S_m$, $\Psi(i) \in G$ and $\sigma^{N-1}(\Psi(i)) = f \circ \alpha(i) \in T$. Also for each $i \in \{0, \ldots, m-1\}$, $d(\Psi(i+1), \Psi(i)) < d(g \circ \beta(i+1), g \circ \beta(i)) + d(f \circ \alpha(i+1), f \circ \alpha(i)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

There is an $i_\epsilon \in S_m$ such that $\frac{1}{2}$ is between $\alpha(i_\epsilon)$ and $\alpha(i_\epsilon + 1)$. It follows that $|\alpha(i_\epsilon) - \frac{1}{2}| < \epsilon_2$. Therefore $d(f \circ \alpha(i_\epsilon), f(\frac{1}{2})) < \epsilon_1$, and thus $|\pi_1 \circ f \circ \alpha(i_\epsilon) - \pi_1(q)| = |\pi_N \circ g \circ \beta(i_\epsilon) - \pi_N(q)| < \epsilon_1$. Thus $d(\Psi(i_\epsilon), q) < \sum_{i=1}^{N} \frac{|\pi_i \circ g \circ \beta(i_\epsilon) - \pi_i(q)|}{2^i} + d(f \circ \alpha(i_\epsilon), f(\frac{1}{2})) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Also there is a j_0 such that $\frac{1}{2}$ is between $\beta(j_0)$ and $\beta(j_0+1)$. Then $d(\Psi(j_0), P') = d(\Psi(j_0), g(\frac{1}{2})) \leq d(g \circ \beta(j_0), g(\frac{1}{2})) + \frac{1}{2^N} < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}$. Thus $\Psi(j_0) \in F \subset G \setminus L$.

Let $\{\epsilon_i\}$ be a sequence of positive numbers less than $\frac{\delta}{4}$ and converging to 0. Construct $\Psi_i : S_{m_i} \to G$ and i_{ϵ_i} as above for each ϵ_i, leaving N and δ fixed, and let W be the closure of $\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{m_i} \Psi_i(j)$. Since for each i and each j in the domain of

Ψ_i, $\sigma^{N-1}(\Psi_i(j)) \in T$, and since σ^{n-1} is continuous and T is closed, $\sigma^{n-1}(W) \subset T$. By Lemma 4, W contains an arc W' with one endpoint q and the other endpoint in $F \subset G \setminus L$. So $L \subset W'$. Therefore $G = \sigma^{N-1}(G) = \sigma^{N-1}(L) \cup \sigma^{N-1}(G \setminus L) \subset T$. So $T = G$. □

Theorem 12. *If G has two Type II points and one of them is contained in A, then G is an arc.*

Proof. Let $\{q_1, q_2\}$ be the set of Type II points in G. Assume $q_2 \in A$. Since q_2 is an endpoint of G, $q_2 \in \{P_0, P_1\}$. Assume $q_2 = P_0$. The proof for $q_2 = P_1$ is the same. Let T be a continuum in G that contains A and is irreducible about the set $\{q_1, P_0, P_1\}$. Let $f : [0,1] \to T$ be a continuous function such that $f(0) = P_0$, $f(1) = P_1$, and $f(\frac{1}{2}) = q_1$ such that for each $x \in [0, \frac{1}{2}]$, $f(x)$ is contained in the arc in T from q_1 to P_0, and for each $x \in [\frac{1}{2}, 1]$, $f(x)$ is contained in the arc in T from q_1 to P_1 . Let L be an arc in $T \setminus \{P_o\}$ that contains q_1 in its interior and such that every point in $L \setminus \{q_1\}$ has order two in G.

If A contains points x and y such that $\pi_n(x) < \pi_n(q_1) < \pi_n(y)$ for some $n \in N$, then A contains a point z_n such that $\pi_n(z_n) = \pi_n(q_1)$. By Lemma 7 $\pi_{1,n}(z_n) = \pi_{1,n}(q_1)$ for each $i \leq n$. So since A is closed q_1 is not in A, there is an $N_1 \in \mathbb{N}$ such that if $n > N_1$then either $\pi_n(x) < \pi_n(q_1)$ for each $x \in A$ or $\pi_n(x) > \pi_n(q_1)$ for each $x \in A$. It follows that for $n \geq N_1$, $\sigma^n(A)$ does not contain both P_1 and P_0. But $\sigma^2(P_o) = P_0$, so $\sigma^m(A)$ must contain P_o for each even m. It follows that for $n \geq N_1$ and n even, P_1 is not in $\sigma^n(A)$.

Since each point of $\overline{G \setminus L}$ has a Type I neighborhood and $\overline{G \setminus L}$ is compact, there is an $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ and $x \in \overline{G \setminus L}$ then $\sigma^{n-1}(x) \in A \subset T$. So there is an $N_3 \in \mathbb{N}$ such that if $n \geq N_3$ and $n-1$ is even then $\sigma^{n-1}(\overline{G \setminus L}) \subset A \setminus \{P_1\}$. Let N be an odd integer such that $N \geq N_3$. Also note that $\sigma^2(q_1) = q_1$ and $\sigma^2(q_2) = q_2$. Therefore $\pi_N(q_1) = \pi_1(q_1)$ and $\pi_N(q_2) = \pi_1(q_2)$. Let R_0 and R_1 be points in G such that $\pi_N(R_0) = 0$ and $\pi_N(R_1) = 1$. And let $g : [0,1] \to G$ be a continuous function such that $g(0) = R_0$, $g(1) = R_1$.

Then $\pi_1 \circ f : [0,1] \to [0,1]$ and $\pi_N \circ g : [0,1] \to [0,1]$ are continuous functions with $\pi_1 \circ f(0) = 0 = \pi_N \circ g(0)$ and $\pi_1 \circ f(1) = 1 = \pi_N \circ g(1)$. Let $\epsilon > 0$ such that $\epsilon < 0$. By Lemma 7 there is an $\epsilon_1 > 0$ such that if $i \in \{1,2\}$ and $y \in G$ such that $|\pi_N(q_i) - \pi_N(y)| < \epsilon_1$ then $\sum\limits_{i=1}^{N} \frac{|\pi_i(q_i) - \pi_i(y)|}{2^i} < \frac{\epsilon}{2}$. Also, there is an $\epsilon_2 > 0$ such that if $s, t \in [0,1]$ and $|s - t| < \epsilon_2$ then $d(f(s), f(t)) < \min\left\{\frac{\epsilon}{2}, \epsilon_1\right\}$ and $d(g(s), g(t)) < \min\left\{\frac{\epsilon}{2}, \epsilon_1\right\}$. According to Lemma 1 there is an $m \in \mathbb{N}$ and functions $\alpha, \beta : S_m \to [0,1]$ such that $\pi_1 \circ f \circ \alpha = \pi_N \circ g \circ \beta$, $\alpha(0) = 0 = \beta(0)$, $\alpha(1) = 1 = \beta(1)$, and for each $i \in \{0, \ldots, m-1\}$, $|\alpha(i+1) - \alpha(i)| < \epsilon_2$ and $|\beta(i+1) - \beta(i)| < \epsilon_2$.

Define $\Psi : S_m \to G$ by $\Psi(i) = \pi_{1,N-1}(g \circ \beta(i)) \oplus f \circ \alpha(i)$. Then for each $i \in S_m$, $\Psi(i) \in G$ and $\sigma^{N-1}(\Psi(i)) = f \circ \alpha(i) \in T$. For each $i \in \{0, \dots, m-1\}$, $d(\Psi(i+1), \Psi(i)) < d(g \circ \beta(i+1), g \circ \beta(i)) + d(f \circ \alpha(i+1), f \circ \alpha(i)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Let $i_\epsilon = \max\left\{i \in S_m \,|\, \alpha(i) \le \frac{1}{2}\right\}$. It follows that $|\alpha(i_\epsilon) - \frac{1}{2}| < \epsilon_2$. Therefore $d(f \circ \alpha(i_\epsilon), f(\frac{1}{2})) < \epsilon_1$, and thus $|\pi_1 \circ f \circ \alpha(i_\epsilon) - \pi_1(q_1)| = |\pi_N \circ g \circ \beta(i_\epsilon) - \pi_N(q_1)| < \epsilon_1$. Thus $d(\Psi(i_\epsilon), q) < \sum_{i=1}^{N} \frac{|\pi_i \circ g \circ \beta(i_\epsilon) - \pi_i(q_1)|}{2^i} + d(f \circ \alpha(i_\epsilon), f(\frac{1}{2})) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Also since $\pi_N(P_0) = \pi_N(q_2) = \pi_1(q_2) = \pi_1 \circ f \circ \alpha(0)) = \pi_N \circ g \circ \beta(0)$, $\pi_{1,N-1}(g \circ \beta(0)) = \pi_{1,N-1}(q_2)$ by Lemma 7 part ii. So $\Psi(0) = \pi_{1,N-1}(g \circ \beta(0)) \oplus f \circ \alpha(0) = \pi_{1,N-1}(q_2) \oplus f(0) = \pi_{1,N-1}(q_2) \oplus q_2 = q_2 = P_o$. Note that for $i \in \{0, \dots, i_\epsilon\}$, $\sigma^{N-1}(\Psi(i)) = f \circ \alpha(i) \in f[0, \frac{1}{2}]$ which is the irreducible arc in T from P_0 to q_1.

Let $\{\epsilon_k\}$ be a sequence of positive numbers converging to 0. Construct $\Psi_k : S_{m_i} \to G$ as above for each ϵ_k, leaving N and δ fixed, and let W be the closure of $\bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i_{\epsilon_k}} \Psi_i(j)$. By Lemma 4, W contains an arc W' with one endpoint q_1 and the other endpoint P_0. Note that $\sigma^{N-1}(W) \subset f([0, \frac{1}{2}])$. Also $L \subset W'$. Therefore $G = \sigma^{N-1}(G) = \sigma^{N-1}(L) \cup \sigma^{N-1}(G \setminus L) \subset T$. Since $T \subset G$, we have that $G = T$. But P_1 is not in $\sigma^{N-1}(G \setminus L)$, so $P_1 \in \sigma^{N-1}(L) \subset f([0, \frac{1}{2}])$. Therefore T is irreducible about $\{q_1, P_0\}$. That is $G = T$ is an arc. □

The final case is not surprisingly the most complicated.

Theorem 13. *If G has two Type II points and neither is contained in A, then G is an arc or a simple triod.*

Proof. Let $\{q_1, q_2\}$ be the set of Type II points in G. Let T be a continuum in G that contains A and is irreducible about the set $\{q_1, q_2, P_0, P_1\}$. Let $f : [0,1] \to T$ be a continuous function such that $f(0) = P_0$, $f(1) = P_1$, $f(\frac{1}{3}) = q_1$, $f(\frac{2}{3}) = q_2$, such that for each $x \in [0, \frac{1}{3}]$, $f(x)$ is contained in the arc in T from P_1 to q_1, for each $x \in [\frac{1}{3}, \frac{2}{3}]$, $f(x)$ is contained in the arc in T from q_1 to q_2, and for each $x \in [\frac{2}{3}, 1]$, $f(x)$ is contained in the arc in T from q_2 to P_1. Let $\delta = d(\{q_1, q_2\}, P_0)$. Let L be a set that is the union of two disjoint arcs such that each arc contains an element of $\{q_1, q_2\}$ in its interior, such that every point in $L \setminus \{q_1, q_2\}$ has order two. Let F be a connected closed subset of $G \setminus L$ such that P_0 is contained in the interior of F. Let $\delta > 0$ be such that if $x \in G$ and $d(x, \{q_1, q_2\}) < \delta$ then $x \in L$ and if $x \in G$ and $d(x, P_0) < \delta$ then $x \in F$.

Next it will shown that there is an $M \in \mathbb{N}$ such that if $n > M$ then $\sigma^n(A) \subset A \setminus \{P_0, P_1\}$. Note that either $\sigma(q_1) = q_1$ and $\sigma(q_2) = q_1$ or $\sigma(q_1) = q_2$. In the first case let $a, b \in [0,1]$ such that $\pi_n(q_1) = a$ and $\pi_n(q_2) = b$ for each $n \in \mathbb{N}$. In the latter case let $a, b \in [0,1]$ such that $\pi_{2n-1}(q_1) = \pi_{2n}(q_2) = a$ and $\pi_{2n}(q_1) = \pi_{2n-1}(q_2) = b$ for each $n \in \mathbb{N}$. Without loss of generality assume $a < b$. For each $n \in \mathbb{N}$ let $v_n, w_n \in G$ such that $\pi_n(v_n) = 0$ and $\pi_n(w_n) = 1$. If A contains an element u

such that $\pi_n(u) < a$ then the arc in G from u to w_n contains a point y_n such that $\pi_n(y_n) = a$ and a point z_n such that $\pi_n(z_n) = b$. So either $\pi_{1,n}(y_n) = \pi_{1,n}(q_1)$ and $\pi_{1,n}(z_n) = \pi_{1,n}(q_2)$ or $\pi_{1,n}(y_n) = \pi_{1,n}(q_2)$ and $\pi_{1,n}(z_n) = \pi_{1,n}(q_1)$. Since q_1 and q_2 are endpoints of G, for sufficiently large n, $d(w_n, q_1) < \frac{1}{2^n}$ and $d(w_n, q_2) < \frac{1}{2^n}$. Thus there is an $M_1 \in \mathbb{N}$ such that if $n > M_1$ and $u \in A$ then $\pi_n(u) > a$. Similarly if A contains an element u such that $\pi_n(u) > b$ then the arc in G from u to v_n contains a point y_n such that $\pi_n(y_n) = a$ and a point z_n such that $\pi_n(z_n) = b$. Therefore, as above, there is an $M_2 \in \mathbb{N}$ such that if $n > M_2$ and $u \in A$ then $\pi_n(u) < b$. It follows that if $n > \max\{M_1, M_2\}$ then $a < \pi_n(x) < b$ for each $x \in A$, and therefore $\sigma^n(A) \subset A \setminus \{P_0, P_1\}$.

Since each point of $\overline{G \setminus L}$ has a Type I neighborhood, and $\overline{G \setminus L}$ is compact, there is an $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ and $x \in \overline{G \setminus L}$ then $\sigma^{n-1}(x) \in A$. So there is an $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $\sigma^{n-1}(\overline{G \setminus L}) \subset A \setminus \{P_0, P_1\}$. Let N be an odd integer such that $N \geq N_2$ and $\frac{1}{2^N} < \frac{\delta}{2}$. Since $\sigma^2(q_1) = q_1$ and $\sigma^2(q_2) = q_2$, $\pi_N(q_1) = \pi_1(q_1)$, and $\pi_N(q_2) = \pi_1(q_2)$. Let R_0 and R_1 be points in G such that $\pi_N(R_0) = 0$ and $\pi_N(R_1) = 1$. And let $g : [0,1] \to G$ be a continuous function such that $g(0) = R_0$, $g(1) = R_1$, and $g(\frac{1}{2}) = P_o$.

Then $\pi_1 \circ f : [0,1] \to [0,1]$ and $\pi_N \circ g : [0,1] \to [0,1]$ are continuous functions with $\pi_1 \circ f(0) = 0 = \pi_N \circ g(0)$ and $\pi_1 \circ f(1) = 1 = \pi_N \circ g(1)$. Let $\epsilon > 0$ such that $\epsilon < \frac{\delta}{2}$. By Lemma 7 there is an $\epsilon_1 > 0$ such that if $y \in G$ and $q \in \{q_1, q_2\}$ such that $|\pi_N(q) - \pi_N(y)| < \epsilon_1$ then $\sum\limits_{i=1}^{N} \frac{|\pi_i(q) - \pi_i(y)|}{2^i} < \frac{\epsilon}{2}$. Also, there is an $\epsilon_2 > 0$ such that if $s, t \in [0,1]$ and $|s - t| < \epsilon_2$ then $d(f(s), f(t)) < \min\left\{\frac{\epsilon}{2}, \epsilon_1\right\}$ and $d(g(s), g(t)) < \min\left\{\frac{\epsilon}{2}, \epsilon_1\right\}$. According to Lemma 1 there is an $m \in \mathbb{N}$ and functions $\alpha, \beta : S_m \to [0,1]$ such that $\pi_1 \circ f \circ \alpha = \pi_N \circ g \circ \beta$, $\alpha(0) = 0 = \beta(0)$, $\alpha(1) = 1 = \beta(1)$, and for each $i \in \{0, \ldots, m-1\}$, $|\alpha(i+1) - \alpha(i)| < \epsilon_2$ and $|\beta(i+1) - \beta(i)| < \epsilon_2$.

Define $\Psi : S_m \to G$ by $\Psi(i) = \pi_{1,N-1} \circ g \circ \beta(i) \oplus f \circ \alpha(i)$. Then for each $i \in S_m$, $\Psi(i) \in G$ and $\sigma^{N-1}(\Psi(i)) = f \circ \alpha(i) \in T$. And for each $i \in \{0, \ldots, m-1\}$, $d(\Psi(i+1), \Psi(i)) < d(g \circ \beta(i+1), g \circ \beta(i)) + d(f \circ \alpha(i+1), f \circ \alpha(i)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

There is a j_0 such that $\frac{1}{2}$ is between $\beta(j_0)$ and $\beta(j_0+1)$. Then $d(\Psi(j_0), P_0) = d(\Psi(j_0), g(\frac{1}{2})) \leq d(g \circ \beta(j_0), g(\frac{1}{2})) + \frac{1}{2^N} < \frac{\delta}{2} + \frac{\delta}{2} = \delta$. Thus $\Psi(j_0) \in F \subset G \setminus L$. Consider the following three cases: case I: $\alpha(j_0) \leq \frac{1}{3}$, case II: $\frac{1}{3} \leq \alpha(j_0) \leq \frac{2}{3}$, and case II: $\frac{2}{3} \leq \alpha(j_0)$.

For case I let $j_1(\epsilon) = j_0$, let $j_2(\epsilon) = \max\left\{i \geq j_0 \,|\, \alpha(k) \leq \frac{1}{3} \text{ for each } k \text{ such that } j_0 \leq k \leq i\right\}$, and let $j_3(\epsilon) = \max\left\{i \geq j_2(\epsilon) \,|\, \alpha(k) \leq \frac{2}{3} \text{ for each } k \text{ such that } j_0 \leq k \leq i\right\}$. It follows that $|\alpha(j_2(\epsilon)) - \frac{1}{3}| < \epsilon_2$ and $|\alpha(j_3(\epsilon)) - \frac{2}{3}| < \epsilon_2$. Therefore $d(f \circ \alpha(j_2(\epsilon)), f(\frac{1}{3})) < \epsilon_1$ and $d(f \circ \alpha(j_3(\epsilon)), f(\frac{2}{3})) < \epsilon_1$. Thus $d(\Psi(j_2(\epsilon)), q_1) < \sum\limits_{i=1}^{N} \frac{|\pi_i \circ g \circ \beta(j_2(\epsilon)) - \pi_i(q_1)|}{2^i} + d(f \circ \alpha(j_2(\epsilon)), f(\frac{1}{3})) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ and $d(\Psi(j_3(\epsilon)), q_2) < \sum\limits_{i=1}^{N} \frac{|\pi_i \circ g \circ \beta(j_3(\epsilon)) - \pi_i(q_2)|}{2^i} +$

$d(f\circ\alpha(j_3(\epsilon)), f(2)) < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$. Note that for each i between $j_1(\epsilon)$ and $j_3(\epsilon)$, $\alpha(i) \leq \frac{2}{3}$, so $\sigma^{N-1}(\Psi(i)) = f\circ\alpha(i) \in f([0,\frac{2}{3}])$.

For case II let $j_1(\epsilon) = \min\{i \leq j_0 \mid \alpha(k) \geq \frac{1}{3}$ for each k such that $i \leq k \leq j_0\}$, let $j_2(\epsilon) = j_0$, and let $j_3(\epsilon) = \max\{i \geq j_0 \mid \alpha(k) \leq \frac{2}{3}$ for each k such that $j_o \leq k \leq i\}$. As in case I we have $d(\Psi(j_1(\epsilon)), q_1) < \epsilon$ and $d(\Psi(j_3(\epsilon)), q_2) < \epsilon$, and for each i between $j_1(\epsilon)$ and $j_3(\epsilon)$, $\frac{1}{3} \leq \alpha(i) \leq \frac{2}{3}$, so $\sigma^{N-1}(\Psi(i)) = f\circ\alpha(i) \in f([\frac{1}{3},\frac{2}{3}])$.

For case III let $j_1(\epsilon) = \min\{i \leq j_0 \mid \alpha(k) \geq \frac{1}{3}$ for each k such that $i \leq k \leq j_0\}$, let $j_2(\epsilon) = \min\{i \geq j_1(\epsilon) \mid \alpha(k) \geq \frac{2}{3}$ for each k such that $j_1 \leq k \leq i\}$, and let $j_3(\epsilon) = j_0$. As in case I and case II, $d(\Psi(j_1(\epsilon)), q_1) < \epsilon$ and $d(\Psi(j_2(\epsilon)), q_2) < \epsilon$, and for each i between $j_1(\epsilon)$ and $j_3(\epsilon)$, $\frac{1}{3} \leq \alpha(i) \leq \frac{2}{3}$, so $\sigma^{N-1}(\Psi(i)) = f\circ\alpha(i) \in f([\frac{1}{3},0])$.

Let $\{\epsilon_i\}$ be a sequence of positive numbers less than $\frac{\delta}{4}$ and converging to 0. Construct $\Psi_i : S_{m_i} \to G$ as above for each ϵ_i, leaving N and δ fixed. Since case I, case II, or case III must hold for each ϵ_i, without loss of generality assume the same case holds for all ϵ_i, and let W be the closure of $\bigcup_{i=1}^{\infty}\bigcup_{j=j_1(\epsilon_i)}^{j_3(\epsilon_i)}\Psi_i(j)$. By Lemma 4, W contains an arc W_1 with one endpoint q_1 and the other endpoint in F, and an arc W_2 with one endpoint q_2 and the other endpoint in F. Either $\sigma^{N-1}(W_1\cup W_2) \subset f([0,\frac{2}{3}])$ or $\sigma^{N-1}(W_1\cup W_2) \subset f([\frac{1}{3},1])$ depending on which of case I, case II, or case III holds for all ϵ_i. Since $L \subset W_1\cup W_2$, $G = \sigma^{N-1}(G) = \sigma^{N-1}(L)\cup\sigma^{N-1}(G\setminus L) \subset T$. Since $T \subset G$, we have that $G = T$. But $\sigma^{N-1}(G\setminus L) \subset T\setminus\{P_0,P_1\}$, and either $\sigma^{N-1}(L) \subset f([0,\frac{2}{3}])$ or $\sigma^{N-1}(L) \subset f([\frac{1}{3},1])$. So either $P_0 \in f([\frac{1}{3},1])$ or $P_1 \in f([0,\frac{2}{3}])$. So either T is irreducible about $\{P_0,q_1,q_2\}$ or T is irreducible about $\{P_1,q_1,q_2\}$. Thus $G = T$ is either an arc or a simple triod. □

Theorem 14. *If $f : [0,1] \to 2^{[0,1]}$ is a surjective upper semi-continuous function such that $\varprojlim f$ is a finite graph, then $\varprojlim f$ is an an arc or a simple triod.*

4. Conclusion

There remains this specific question.

Question. Is there a surjective upper semi-continuous function $f : [0,1] \to 2^{[0,1]}$ such that $\varprojlim f$ is a simple triod?

There is also the more general question of what topological properties do continua that are the inverse limit with a single closed subset of $[0,1]\times[0,1]$ share. This collection includes all chainable continua that are the inverse limit with a single continuous bonding map and as far as we know an ad hoc collection of dendroids, dendrites, and some continua that contain cycles, but no finite graphs except an arc and maybe a triod. Inverse limits with set valued functions can be useful for producing a complicated continuum from a simple subset of $[0,1]\times[0,1]$ and for studying the dynamics of set valued functions, but the more general information we have about what topological properties are shared by these inverse limits the more powerful the tool they become.

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