INVERSE LIMITS WITH SET VALUED FUNCTIONS

VAN NALL

Abstract. We begin to answer the question of which continua in the Hilbert cube can be the inverse limit with a single upper semi-continuous bonding map from $[0,1]$ to $2^{[0,1]}$. Several continua including $[0,1] \times [0,1]$ and all compact manifolds with dimension greater than one cannot be equal to such an inverse limit. It is also shown that if the upper semi-continuous bonding maps have only zero dimensional point values, then the dimension of the inverse limit does not exceed the dimension of the factor spaces.

1. Introduction

The principle advantage of the inverse limit approach in the study of continua is that a very complicated continuum can be described in terms of a single simple function. Recent work by Mahavier and Ingram [2] has raised interest in inverse limits with upper semi-continuous set valued functions. An important question with all inverse limits is what structures of the inverse limit are determined by the factor spaces and the bonding maps. This paper considers this question with the focus on dimension and primarily on inverse limits using a single set valued bonding map on one dimensional factor spaces. For example, it is known that the inverse limit with a single set valued function from an arc to an arc can have any finite dimension or even be infinite dimensional [2, Example 5, p. 129]. So we ask what sort of set valued functions yield inverse limits with dimension higher than their factor spaces.

This work was motivated by a more specific question asked by Tom Ingram. Is there an upper semi-continuous set valued function f from $[0,1]$ into $2^{[0,1]}$ such that the inverse limit with the single function f is homeomorphic to a 2-cell? It will be shown that if X is any continuum such that every nonempty subcontinuum of X contains an open set, then the inverse limit with a single upper semi-continuous set valued function from a continuum X into 2^X is not homeomorphic to an n dimensional manifold with n bigger than one. In addition it will be shown that if the upper semi-continuous set valued functions do not have a value at a point with

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dimension one or more, then the dimension of the inverse limit cannot be higher than the dimension of the factor spaces.

2. Definitions and Notation

In this paper all spaces are separable and metric. A continuum is a compact and connected separable metric space. If $\{X_i\}$ is a countable collection of compact spaces each with metric d_i such that $diam(X_i) \leq 1$ for each i, then $\Pi_{i=1}^{\infty} X_i$ represents the countable product of the collection $\{X_i\}$, with metric given by $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$. Note that in this article a sequence will be denoted with bold type and the terms of the sequence in italic type so that, for example, $\mathbf{x} = (x_1, x_2, x_3, \ldots)$. For each i let $\pi_i : \Pi_{i=1}^{\infty} X_i \to X_i$ be defined by $\pi_i(\mathbf{x}) = \pi_i((x_1, x_2, x_3, \ldots)) = x_i$. For each i, let $f_i : X_{i+1} \to 2^{X_i}$ be a set valued function where 2^{X_i} is the space of closed subsets of X_i with the Hausdorff metric. The inverse limit of the sequence of pairs $\{(f_i, X_i)\}$, denoted $\varprojlim(f_i, X_i)$, is defined to be the set of all $(x_1, x_2, x_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$ such that $x_i \in f_i(x_{i+1})$ for each i. The functions f_i are called bonding maps, and the spaces X_i are called factor spaces. A set valued function $f : X \to 2^Y$ into the closed subsets of Y is upper semi-continuous (usc) if for each open set $V \in Y$ the set $\{x : f(x) \subset V\}$ is an open set in X. For the countable product of a single space X define the shift map $\sigma : \Pi_{i=1}^{\infty} X \to \Pi_{i=1}^{\infty} X$ by $\sigma((x_1, x_2, x_3, \ldots)) = (x_2, x_3, x_4, \ldots)$. We say a subset $M \subseteq \Pi_{i=1}^{\infty} X$ is shift invariant if $\sigma(M) = M$. In this paper $dim(X)$ refers to the covering dimension. That is, for a compact set X, $dim(x) \leq n$ if and only if for each $\epsilon > 0$ there is an open cover of X with mesh less than ϵ and order less than or equal to n, where the order of a cover is the largest integer n such that there are $n + 1$ members of the cover which have non-empty intersection [1, p. 67].

If $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$ and $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$ and $x_i = y_i$ for some i, then define $Cr_i(\mathbf{x}, \mathbf{y}) = (x_1, x_2, \ldots, x_i, y_{i+1}, y_{i+2}, \ldots)$. Note then that $Cr_i(\mathbf{x},\mathbf{x}) = \mathbf{x}$ for each $\mathbf{x} \in M$ and each i. For a subset M of $\Pi_{i=1}^{\infty} X$, let $Cr(M)$ be the set of all $\mathbf{z} \in \Pi_{i=1}^{\infty} X_i$ such that there is an i and elements $\mathbf{x}$ and $\mathbf{y}$ in M such that $\mathbf{z} = Cr_i(\mathbf{x}, \mathbf{y})$. Note that $M \subseteq Cr(M)$. We say a subset $M \subseteq \Pi_{i=1}^{\infty} X_i$ contains all crossovers if $Cr(M) = M$. It is easy to verify that if there are functions $f_i : X_{i+1} \to 2^{X_i}$, and $M = \varprojlim(f_i, X_i)$, then $Cr(M) = M$.

Finally, a subcontinuum A of a continuum X is a free arc if A is an arc such that the boundary of A is contained in the set of endpoints of A, and a continuum X is a finite graph if X is the union of a finite number of free arcs.

3. Upper Semi-continuity, Compact Inverse Limits, and Crossovers

In [2, Theorem 2.1, p. 120] it is shown that if $f : X \to 2^Y$ is a set valued function and X and Y are compact, then f is usc if and only if the graph of f is closed. It follows that if $g : X \to Y$ is a continuous function between compact sets X and Y, and $Y = g(X)$, then $g^{-1} : Y \to 2^X$ is an usc set valued function. It is an elementary exercise to show that if X, Y, and Z are compact, then the composition of two usc functions $g : X \to 2^Z$ and $f : Z \to 2^Y$is a usc function provided what is meant by the composition $f \circ g$ is the set of all pairs (x, y) such that there is a $z \in g(x)$ such that $y \in f(z)$.

Theorem 3.1. *If $M = \varprojlim(f_i, X_i)$ where X_i is compact, $f_i : X_{i+1} \to 2^{X_i}$, and $\pi_i(M) = X_i$ for each i, then M is compact if and only if each f_i is usc.*

Proof. Assume $M = \varprojlim(f_i, X_i)$, where X_i is compact, and $f_i : X_{i+1} \to 2^{X_i}$. If each f_i is usc then M is compact by [2, Theorem 3.2 p.121] . So suppose M is compact. Let $p_i = \pi_i|_M$. Then $f_i = p_i \circ p_{i+1}^{-1}$ for each i. Since each f_i is the composition of usc functions, each f_i is usc. □

Theorem 3.2. *Suppose each X_i is a compact space, and M is a compact subset of $\Pi_{i=1}^{\infty} X_i$, and $X_i' = \pi_i(M)$ for each i, then the following are equivalent:*

i. *There exist set valued functions $g_i : X_{i+1}' \to 2^{X_i'}$ such that $M = \varprojlim(g_i, X_i')$.*
ii. *There exist usc set valued functions $f_i : X_{i+1}' \to 2^{X_i'}$ such that $M = \varprojlim(f_i, X_i')$.*
iii. *M contains all crossovers.*

Proof. Assume each X_i is a compact space and M is a compact subset of $\Pi_{i=1}^{\infty} X_i$, and $X_i' = \pi_i(M)$ for each i. That property ii follows from property i is established in the previous theorem. It is easy to see that an inverse limit contains all crossovers, so property ii imples property iii. So all that remains is to show that if M contains all crossovers, then there exist usc set valued functions $f_i : X_{i+1}' \to 2^{X_i'}$ such that $M = \varprojlim(f_i, X_i')$.

Assume M contains all crossovers. Let $p_i = \pi_i|_M$. Define the set valued function $f_i : X_{i+1}' \to 2^{X_i'}$ by $f_i = p_i \circ p_{i+1}^{-1}$. Since each f_i is the composition of usc set valued functions, each f_i is usc. If $(x_1, x_2, x_3, \ldots) \in M$, then $x_i \in p_i \circ p_{i+1}^{-1}(x_{i+1}) = f_i(x_{i+1})$ for each i. So $M \subseteq \varprojlim(f_i, X_i')$. Now assume $(x_1, x_2, x_3, \ldots) \in \varprojlim(f_i, X_i')$. For each i, since $x_i \in f_i(x_{i+1}) = p_i \circ p_{i+1}^{-1}(x_{i+1})$, there is a $\mathbf{z} \in M$ such that $p_i(\mathbf{z}) = x_i$, and $p_{i+1}(\mathbf{z}) = x_{i+1}$. Since M contains all crossovers it follows that for each n there is a $\mathbf{z}^n \in M$ such that $p_i(\mathbf{z^n}) = x_i$ for each $i \leq n$. The sequence $\{\mathbf{z}^n\}$ converges to $(x_1, x_2, x_3, \ldots)$, and, since M is closed, $(x_1, x_2, x_3, \ldots) \in M$. □

4. Inverse Limits with a Single Set Valued Function

If $G \subset X \times X$ with $\pi_1(G) = \pi_2(G) = X$, then G is closed if and only if G is the graph of a usc function $f : X \to 2^X$ [2, Theorem 2.1 p. 120]. We will use $\varprojlim G$ to refer to the inverse limit of the single function from X to 2^X whose graph is G and in so doing will always imply that $\pi_1(G) = \pi_2(G) = X$. Let $I = [0, 1]$. An example is given in [2, Example 5, p. 129] of a closed subset G of $I \times I$ so that $\underleftarrow{lim} G$ is homeomorphic to $I \times I \cup ([-1, 0] \times \{0\})$. So it is natural to ask if there is a closed subset $G \subset I \times I$ such that $\underleftarrow{lim} G$ is homeomorphic to $I \times I$. In light of Theorem 3.2 this is equivalent to asking if the Hilbert cube contains a subset K homeomorphic to $I \times I$ such that $\sigma(K) = K$ and $Cr(K) = K$. In this section we show that there are many different continua, all having dimension at least two, including $I \times I$, that are not homeomorphic to a subset K of the Hilbert cube such that $\sigma(K) = K$ and $Cr(K) = K$. The question remains whether there is a one dimensional continuum that is homeomorphic to a subset K of the Hilbert cube such that $\sigma(K) = K$ and $Cr(K) = K$.

The proof of the following theorem depends at a crucial point on the following observations about the shift map. First, if $M = \underleftarrow{lim} G$ for some closed $G \subset X \times X$, then $\sigma(M) = M$. Second, if K is a compact subset of M such that $\sigma|_{\mathrm{K}}$ is one-to-one, then K and $\sigma(K)$ are homeomorphic subsets of M. In particular, if K is a compact subset of M such that $\pi_1(K)$ is a singleton set, then K and $\sigma(K)$ are homeomorphic subsets of M. Finally, we say a collection $\{K_\alpha\}_{\alpha \in \Lambda}$ of continua has the finite chain property if for each pair of points $\{x, y\} \subset \bigcup\limits_{\alpha \in \Lambda} K_\alpha$ there is a finite subcollection $\{K_{\alpha_1}, K_{\alpha_2}, \ldots, K_{\alpha_n}\}$ of $\{K_\alpha\}_{\alpha \in \Lambda}$ such that $x \in K_{\alpha_1}$, $y \in K_{\alpha_n}$, and $K_{\alpha_i} \cap K_{\alpha_{i+1}} \neq \emptyset$ for $i = 1, 2, \ldots, n-1$.

Theorem 4.1. *Suppose X is a continuum such that every non-degenerate subcontinuum of X contains a nonempty set that is open relative to X. Suppose $n > 1$, and M is an n dimensional continuum that is the union of a countable collection $\{K_i\}$ of continua with the finite chain property such that for each i, every open subset of K_i has dimension n, every compact n dimensional subset of K_i has nonempty interior, and there is no uncountable pairwise disjoint collection of nonempty zero dimensional subsets of K_i each of which separates K_i. Then there is no closed set $G \subset X \times X$ with $\pi_1(G) = \pi_2(G) = X$ such that $M = \underleftarrow{lim} G$.*

Proof. Assume the hypotheses are true for X and M, and assume that $M = \underleftarrow{lim} G$ where $G \subset X \times X$ such that $\pi_1(G) = \pi_2(G) = X$. For each i, let p_i be the restriction to M of the i^{th} projection of $\Pi_{i=1}^{\infty} X$ onto X.

If a closed subset C of M separates M, and C does not separate K_i for each i, then there are integers i and j such that $K_i \cap K_j \neq \emptyset$, and $K_i \cap K_j \subseteq C$. To see this, suppose A and B are nonempty disjoint open sets such that $M \setminus C = A \cup B$. Let $x \in A$, and $y \in B$, and Let $\{K_{n_1}, K_{n_2}, \ldots K_{n_m}\}$ be a finite collection from $\{K_i\}$ such that $x \in K_{n_1}$, $y \in K_{n_m}$, and $K_{n_i} \cap K_{n_{i+1}} \neq \emptyset$ for $i = 1, 2, \ldots m-1$. Let s be the smallest integer such that $K_{n_s} \cap B \neq \emptyset$. Since C doest not separate either

K_{n_s} or K_{n_s-1} it follows that $K_{n_s} \subset B \cup C$ and $K_{n_s-1} \subset A \cup C$, and therefore $K_{n_s-1} \cap K_{n_s} \subset C$.

So if there is an uncountable pairwise disjoint collection $\mathcal{C}$ of closed subsets of M that separate M, there is an i such that for uncountably many $C \in \mathcal{C}$, $C \cap K_i$ separates K_i. It follows that there is no uncountable pairwise disjoint collection of nonempty zero dimensional closed subsets of M each of which separates M.

Since X does not contain a continuum of convergence, X is hereditarily locally connected [3, p. 167]. Therefore X is arc connected [3, p. 130], and therefore X contains a nondegenerate arc, and that arc contains a free arc D. The free arc D contains an uncountable pairwise disjoint collection of pairs of points $\{a_\alpha, b_\alpha\}_{\alpha\in\Lambda}$ each of which separates X. Therefore $p_1^{-1}(\{a_\alpha, b_\alpha\})$ is a closed set that separates M for each $\alpha \in \Lambda$. Therefore, according to the previous paragraph, $dim(p_1^{-1}(\{a_\alpha, b_\alpha\})) > 0$ for uncountably many $\alpha \in \Lambda$. Since $p_1^{-1}(\{a_\alpha, b_\alpha\}) = p_1^{-1}(a_\alpha) \cup p_1^{-1}(b_\alpha)$ for each α, it follows that there exists an uncountable subset A of X such that $dim(p_1^{-1}(x)) > 0$ for each $x \in A$.

Now $p_1^{-1}(x)$ is compact, so $p_1^{-1}(x)$ contains a nondegenerate continuum L for each $x \in A$ [1, p. 22]. This continuum L must have at least one projection that contains a nondegenerate continuum, though $p_1(L) = \{x\}$. So there is an earliest projection $p_m(L)$ that contains a nondegenerate subcontinuum J of X, and each $p_i(L)$ for $i < m$ is a singleton set $\{x_i\}$. Let $L^* = [\bigcap_{i=1}^{m-1} p_i^{-1}(x_i)] \cap p_m^{-1}(J)$. Then $L^* \subseteq p_1^{-1}(x)$, and $p_m(L^*) = J$. If $(z_1, z_2, z_3, \ldots) \in p_1^{-1}(J)$, then, since M contains all crossovers, $(x_1, x_2, \ldots, x_{m-1}, z_1, z_2, \ldots) \in L^*$. Therefore $(z_1, z_2, z_3, \ldots) \in p_1^{-1}(J) \in \sigma^{m-1}(L^*)$. So $p_1^{-1}(J) \subseteq \sigma^{m-1}(L^*)$. But $\sigma^{m-1}(L^*)$ is homeomorphic to L^*. It follows that $p_1^{-1}(x)$ contains a set homeomorphic to $p_1^{-1}(J)$. Since J has nonempty interior in X, the set $p_1^{-1}(J)$ has nonempty interior in M. According to the Baire Category Theorem there is at least one i such that $p_1^{-1}(J) \cap K_i$ has nonempty interior, and therefore $p_1^{-1}(J) \cap K_i$ has dimension n. So $p_1^{-1}(x)$ has dimension n. Since an n dimensional space cannot be the countable union of closed subsets with dimension less than n [1, p. 30], there is at least one j such that $p_1^{-1}(x) \cap K_j$ has dimension n, and therefore $p_1^{-1}(x) \cap K_j$ has nonempty interior in K_j. Since A is uncountable there is a j such that K_j contains an uncountable pairwise disjoint collection of sets with nonempty interior in K_j. But K_j is contained in $\Pi_{i=1}^{\infty} X$ which has a countable basis. This is a contradiction. Therefore there is no closed set $G \subset X \times X$ with $\pi_1(G) = \pi_2(G) = X$ such that $M = \underleftarrow{lim} G$. □

If K is either a closed n-cell or an n dimensional manifold, then K can not be separated by a set with dimension less than $n - 1$[1, p. 46], and a necessary and sufficient condition for a subset of K to have dimension n is for that subset to contain a nonempty set that is open in K [1, p. 48]. So if there is an $n > 1$such that M is a continuum that is the union of a countable collection of closed n-cells and compact n dimensional manifolds with the finite chain property, and X is a finite graph, then the hypothesis of Theorem 4.1 are satisfied by X and M.

Corollary 4.2. *If $n > 1$, and M is a continuum that is the union of a countable collection with the finite chain property of closed n-cells and compact n dimensional manifolds, and X is a finite graph, then there does not exist a closed set $G \subset X \times X$ such that $M = \underleftarrow{lim} G$.*

The following example demonstrates that containing all crossovers is essential in the proof of Theorem 4.1.

Example 4.3. If $h : [\frac{1}{2}, 1] \times [0, 1] \to I^\omega$ is defined by $h(x, y) = (xy, x(1-y), xy, x(1-y), \ldots)$ then it is easy to check that h is a homeomorphism and $M = h([\frac{1}{2}, 1] \times [0, 1])$ is shift invariant. If $x = h(1, \frac{2}{3}) = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \ldots)$ and $y = h(\frac{1}{2}, \frac{1}{3}) = (\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \ldots)$, then $Cr_2(x, y) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \ldots)$ which is not in M. So M is a shift invariant subset of the Hilbert cube which is homeomorphic to $[0, 1] \times [0, 1]$, and, as is required by Theorem 4.1, $Cr(M) \neq M$.

5. Inverse Limits with Higher Dimension than the Factor Spaces

As noted above the inverse limit with the nterval $[0, 1]$ as the factor space and usc set valued functions can have dimension two, and in fact it can have any finite dimension or be infinite dimensional. However, it follows from the theorem below that if the dimension of $f_i(x)$ is zero for each i and for each $x \in [0, 1]$, then $\underleftarrow{lim}(f_i, [0, 1])$ has dimension no greater than one. So, for example, if $G \subset [0, 1] \times [0, 1]$ is the union of the graphs of finitely many continuous functions from $[0, 1]$ to $[0, 1]$ then $\underleftarrow{lim} G$ is one dimensional.

Let $\{X_i\}$be a collection of compact spaces with $dim(X_i) \leq m$ for each i. Suppose $M = \underleftarrow{lim}(f_i, X_i)$ where each $f_i : X_{i+1} \to 2^{X_i}$ is a usc set valued function. For each $n > 1$ let $P_j = X_1 \times X_2 \times \ldots \times X_j$, and define $F_n : X_{n+1} \to 2^{P_n}$ by

$$F_n(x) = \{(x_1, x_2, \ldots, x_n) \in P_n | x_n \in f_n(x), \text{ and } x_i \in f_i(x_{i+1}) \text{ for } 1 \leq \text{i} \leq \text{n} - 1\}$$

Lemma 5.1. *F_n is usc for each n.*

Proof. According to [2, Theorem 2.1, p.120] if the graph of F_n is closed then F_n is usc. Let $G = \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in P_{n+1} | x_i \in f_i(x_{i+1})\}$. Then G is homeomorphic to the graph of F_n. Suppose $(z_1, z_2, \ldots, z_n, z_{n+1}) \in P_{n+1} \setminus G$. Then $z_i \in X_i \setminus f_i(x_{i+1})$ for some $i < n + 1$. Thus (z_i, z_{i+1}) is not in G_i the graph of f_i, which is a closed subset of $X_i \times X_{i+1}$ since f_i is usc [2, Theorem 2.1, p.120]. So there exist open sets V and U such that $(z_i, z_{i+1}) \in V \times U \subseteq X_i \times X_{i+1} \setminus G_i$.

Then $(z_1, z_2, \ldots, z_n, z_{n+1}) \in \pi_i^{-1}(V) \cap \pi_{i+1}^{-1}(U)$, $\pi_i^{-1}(V) \cap \pi_{i+1}^{-1}(U)$ is open is P_n, and $\pi_i^{-1}(V) \cap \pi_{i+1}^{-1}(U) \subseteq P_{n+1} \setminus G$. So G is closed, and therefore F_n is usc. □

Lemma 5.2. *If $dim(F_n(x)) > 0$ for some $n > 1$ and $x \in X_{n+1}$, then there is an $i \leq n+1$ and $z \in X_{i+1}$ such that $dim(f_i(z)) > 0$.*

Proof. Suppose $dim(F_n(x)) > 0$ for some $n > 1$and $x \in X_{n+1}$. Since $F_n(x)$ is compact it must contain a non-degenerate continuum K [1, p. 22]. Let j be the largest integer less than $n+1$ such that $dim(\pi_j(K)) > 0$. If $j = n$, then $\pi_j(K) \subseteq f_n(x)$. So $dim(f_n(x)) > 0$. If $j < n$, then $\pi_{j+1}(K)$ is zero dimensional and connected. So $\pi_{j+1}(K) = \{z\}$ for some $z \in X_{j+1}$, and $\pi_j(K) \subseteq f_j(z)$. So $dim(f_j(z)) > 0$. □

Theorem 5.3. *If $M = \underleftarrow{lim}(f_i, X_i)$ where each X_i is a compact space with $dim(X_i) \leq m$, and each f_i is a usc set valued function such that $dim(f_i(x)) = 0$ for each i , and each $x \in X_{i+1}$, then $dim(M) \leq m$.*

Proof. Assume $diam(X_i) \leq 1$ for each i. Let $\epsilon > 0$, and let n be such that $\sum\limits_{i=n+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2}$, and let $\delta > 0$ be such that $\sum\limits_{i=1}^{n} \frac{\delta}{2^i} < \frac{\epsilon}{2}$. By Lemma 5.2 $dim(F_{n-1}(x)) = 0$ for each $x \in X_n$. For each $x \in X_n$, let $\mathcal{V}_x$ be a finite pairwise disjoint open cover of $F_{n-1}(x)$ such that $diam(\pi_i(v)) < \delta$ for each $v \in \mathcal{V}_x$ and each $i < n$. Since F_{n-1} is usc, each $x \in X_n$ is contained in an open set U_x such that $F_{n-1}(U_x) \subseteq \bigcup \mathcal{V}_x$. Since $dim(X_n) \leq m$, and X_n is compact, there is a finite open refinement $\mathcal{U}$ of $\{U_x\}_{x \in X_n}$ with mesh δ and order less than $m+1$ covering X_n.

For each $v \in \mathcal{V}_x$ let $v^* = \{(x_1, x_2, x_3, \ldots) \in M | (x_1, x_2, \ldots, x_{n-1}) \in v\}$, and let $\mathcal{V}_x^* = \{v^* | v \in \mathcal{V}_x\}$. Then $\mathcal{V}_x^*$ is a pairwise disjoint collection of open sets in M such that $diam(\pi_i(v^*)) < \delta$ for each $v^* \in \mathcal{V}_x^*$ and each $i < n$.

For each element $u \in \mathcal{U}$ there is an $x \in X_n$ such that $u \subseteq U_x$, and therefore $F_{n-1}(u) \subseteq \bigcup \mathcal{V}_x$. It follows that $\pi_n^{-1}(u) \subseteq \bigcup \mathcal{V}_x^*$. So the collection $\left\{v^* \cap \pi_n^{-1}(u) | v^* \in \mathcal{V}_x^*\right\}$ is a partition of $\pi_n^{-1}(u)$ into pairwise disjoint open sets with diameter less than ϵ. Since the order of the cover $\left\{\pi_n^{-1}(u) | u \in \mathcal{U}\right\}$ of M is the same as the order of $\mathcal{U}$, there is an open cover of M with order less than $m+1$ and mesh ϵ. □

Theorem 5.4. *If X_1 is a continuum such that every nondegenerate subcontinum K of X_1 contains a countable set that separates K, and for each i, X_i is compact, and $f_i : X_{i+1} \to 2^{X_i}$ is usc, and for each $y \in X_i$, $dim(\{x \in X_{i+1} | y \in f_i(x)\}) = 0$, then $dim(\underleftarrow{lim}(f_i, X_i)) \leq 1$.*

Proof. Suppose X_1 is a continuum such that every nondegenerate subcontinum K of X_1 contains a countable set that separates K, and for each i, X_i is compact, and $f_i : X_{i+1} \to 2^{X_i}$ is usc, and for each $y \in X_i$, $dim(\{x \in X_{i+1} | y \in f_i(x)\}) = 0$. Let $M = \underleftarrow{lim}(f_i, X_i)$.

If $z \in X_1$ and $dim(\pi_1^{-1}(z)) > 0$, then $\pi_1^{-1}(z)$ contains a nondegenerate continuum L. Let m be the smallest integer such that $dim(\pi_m(L)) > 0$. Let $\{y\} = \pi_{m-1}(L)$, and then $\pi_m(L) \subset \{x \in X_m | y \in f_{m-1}(x)\}$. This contradicts the assumption that $\{x \in X_m | y \in f_{m-1}(x)\}$ is zero dimensional. So $dim(\pi_1^{-1}(z)) = 0$ for each $z \in X_1$.

If $dim(M) > 1$, then, since M is compact, M contains a continuum K such that every subset of K that separates K has dimension at least one [1, Theorem VI 8, p. 94]. Now $dim(\pi_1^{-1}(x)) = 0$ for each $x \in X_1$. So $\pi_1(K)$ is a nondegenerate subcontinuum of X_1, and therefore $\pi_1(K)$ contains a countable set A that separates $\pi_1(K)$. Therefore $\pi_1^{-1}(A)$ separates K. But $\pi_1^{-1}(A)$ is the countable union of compact zero dimensional sets. So $\pi_1^{-1}(A)$ is zero dimensional. This is a contradiction. Therefore $dim(M) \leq 1$. □

6. Problems

It is not hard to find usc set valued functions on $[0,1]$ whose inverse limit yields a continuum in the Hilbert cube homeomorphic to $[0,1] \times [0,1]$. Such a copy of $[0,1] \times [0,1]$ would contain all crossovers, but according to Theorem 4.1 it cannot be shift invariant. Example 4.3 shows that $[0,1] \times [0,1]$ can be embedded in the Hilbert cube so that it is shift invariant. But then it does not contain all crossovers.

Problem 6.1. Is there a one dimensional continuum that can not be embedded in the Hilbert cube so that it is shift invariant and contains all crossovers? (W. T. Ingram has asked if a simple triod can be embedded in the Hilbert cube so that it is shift invariant and contains all crossovers.)

Problem 6.2. Is there a nondegenerate continuum X such that $X \times [0,1]$ can be embedded in the Hilbert cube so that it is shift invariant and contains all crossovers?

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