CONNECTED INVERSE LIMITS WITH A SET VALUED FUNCTION

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1. Introduction

Inverse limits have been used by topologists for decades to study continua. More recently inverse limits have begun to play a role in dynamical systems, at least among continua theorists who are interested in the role that the topological structure of attractors, orbit spaces, or Julia sets play in the dynamics generated by continuous functions between compact spaces. Also recently Mahavier[3] introduced the study of inverse limits with set valued functions on intervals, and later Mahavier and Ingram [2] generalized to set valued functions on compact sets. There is a growing body of research into the structure of these generalized inverse limits. It has even been suggested that they too could play a role in the study of dynamical systems. That may be, but since we are at the beginning of the study of generalized inverse limits there are some very basic things that need to be better understood. For example, with continuous functions defined between one dimensional continua the resulting inverse limit is a one dimensional continuum. In the case of generalized inverse limits it is possible to have a set valued function between intervals with a one dimensional graph such that the inverse limit with this function is infinite dimensional, and it is possible to have a set valued function between intervals with a connected graph that yields an inverse limit that is not connected. In fact Greenwood and Kennedy have shown [5] that in the collection of all sets that are generalized inverse limits with bonding maps whose graphs are closed connected subsets of $[0,1]\times[0,1]$, those sets that are homeomorphic to the Cantor set form a G_δ set. In addition we do not have general criteria for determining whether or not a given set valued function will produce the relatively rare occurance of a connected generalized inverse limit. Indeed, it looks like such a set of criteria would be very complicated. Our response will be to take a constructive approach to the problem of connected generalized inverse limits. That is our goal is to provide techniques to build set valued functions whose resulting inverse limits will be connected. For example we consider questions like: If $\varprojlim f$ is connected then what sorts of sets can

Date: January 11, 2011.
2000 *Mathematics Subject Classification.* 54C60, 54B10, 54D80.
Key words and phrases. generalized inverse limits, set valued functions, connected.

be added to the graph of f to yield a set valued function g such that $\lim\limits_{\leftarrow} g$ is still connected?

2. Definitions and Notation

A continuum is a compact and connected Hausdorff space. If $\{X_i\}$ is a countable collection of compact spaces, then $\Pi_{i=1}^{\infty} X_i$ represents the countable product of the collection $\{X_i\}$, with the usual product topology. Elements of this product will be denoted with bold type, and the coordinates of the element in italic type, so that, for example, $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$. For each i let $\pi_i : \Pi_{i=1}^{\infty} X_i \to X_i$ be defined by $\pi_i(\mathbf{x}) = \pi_i((x_1, x_2, x_3, \ldots)) = x_i$. The same notation will be used in the case of $\Pi_{i=1}^{n} X_i$, that is $\pi_i : \Pi_{i=1}^{n} X_i \to X_i$ is defined by $\pi_i(\mathbf{x}) = \pi_i((x_1, x_2, x_3, \ldots, x_n)) = x_i$. Also, for $1 \le j < k \le n$, $\pi_{j,k} : \Pi_{i=1}^{n} X_i \to \Pi_{i=j}^{k} X_i$ is defined by$\pi_{j,k}((x_1, x_2, x_3, \ldots, x_n)) = (x_j, x_{j+1}, \ldots, x_k)$.

For each i, let $f_i : X_{i+1} \to 2^{X_i}$ be a set valued function where 2^{X_i} is the hyperspace of compact subsets of X_i. The inverse limit of the sequence of pairs $\{(f_i, X_i)\}$, denoted $\lim\limits_{\leftarrow}(f_i, X_i)$, is defined to be the set of all $(x_1, x_2, x_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$ such that $x_i \in f_i(x_{i+1})$ for each i. The functions f_i are called bonding maps, and the spaces X_i are called factor spaces. The notation $\lim\limits_{\leftarrow} f_i$ will also be used for $\lim\limits_{\leftarrow}(f_i, X_i)$ when the sets X_i are understood, and the notation $\lim\limits_{\leftarrow} G_i$ will sometimes be used for $\lim\limits_{\leftarrow} f_i$ when G_i is the graph of f_i. In this paper we will work exclusively with the case where there is a single set valued function f from a continuum X into 2^X, and $\lim\limits_{\leftarrow} f = \lim\limits_{\leftarrow} f_i$ where $f_i = f$ for each i.

A set valued function $f : X \to 2^Y$ into the compact subsets of Y is upper semi-continuous (usc) if for each open set $V \subset Y$ the set $\{x : f(x) \subset V\}$ is an open set in X. A set valued function $f : X \to 2^Y$ where X is Hausdorff and Y is compact is upper semi-continuous if and only if the graph of f is compact in $X \times Y$ [2, Theorem 4, p. 58]. It is therefore easy to see that if $f : X \to 2^Y$ is upper semi-continuous and X and Y are compact Hausdorff spaces, and G is the graph of f, then the set valued function f^{-1} which has graph $G^{-1} = \{(y, x) : (x, y) \in G\}$ is also upper semi-continuous from Y to 2^X. A set valued function $f : X \to 2^Y$ will be called surjective if for each $y \in Y$ there is a point $x \in X$ such that $y \in f(x)$. In this paper we are only considering inverse limits with a single bonding map and we mean for that assumption to imply that $\pi_{i,i+1}(\lim\limits_{\leftarrow} f)$ is homeomorphic to the graph of f for each i. For that reason it is essential to require that the map f be surjective. Finally, for a fixed continuum X and integers m and n the symbol $\oplus$ represents the binary operation $\oplus : \Pi_{i=1}^{n} X \times \Pi_{i=1}^{m} X \to \Pi_{i=1}^{m+n} X$ defined by $(x_1, x_2, x_3, \ldots, x_n) \oplus (y_1, y_2, y_3, \ldots, y_m) = (x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_n)$.

3. Results

It is easy to construct a set valued function with a connected graph whose composition with itself has a disconnected graph. Since the graph of the composition of the function with itself is homeomorphic to the projection of the inverse limit with this function into the first and third coordinates, such an inverse limit would not be connected. A very simple example of this type is given below.

Before the first example we present a couple of theorems that can be used to show the connectivity of a large class of inverse limits. The first is a generalization of results of Ingram[4, Theorems 3.3 and 4.2]. It is known that a surjective continuum valued upper semi-continuous function from a continuum X to 2^X yields a connected inverse limit [1, Theorem 4.7]. So we want to know when the inverse limit with a function that is the union of continuum valued functions is connected. The following is the most general possible union theorem for this type of function in the sense that the most general union theorem must require that union be closed so that the resulting map is upper semi-continuous, the most general union theorem must require that the union be connected since the graph of the function used to form the inverse limit is a continuous projection of the inverse limit, and finally the restriction to surjective set valued functions was explained earlier, so the most general union theorem should require that the union is the graph of a surjective function.

Theorem 1. *Suppose X is a compact metric space, and $\{F_\alpha\}_{\alpha\in\Lambda}$ is collection of closed subsets of $X \times X$ such that for each $x \in X$ and each $\alpha \in \Lambda$ the set $\{y \in X \ : \ (x, y) \in F_\alpha\}$ is nonempty and connected, and such that $F = \bigcup_{\alpha\in\Lambda} F_\alpha$ is a closed connected subset of $X\times X$ such that for each $y \in X$ the set $\{x \in X \ : \ (x, y) \in F\}$ is nonempty. Then $\underleftarrow{\lim} F$ is connected.*

Proof. Assume X is a compact metric space and $\{F_\alpha\}_{\alpha\in\Lambda}$ is collection of closed subsets of $X\times X$ such that for each $x \in X$ and each $\alpha \in \Lambda$ the set $\{y \in X \mid (x, y) \in F_\alpha\}$ is nonempty and connected, and such that $F = \bigcup_{\alpha\in\Lambda} F_\alpha$ is a closed connected subset of $X \times X$ such that for each $y \in X$ the set $\{x \in X \mid (x, y) \in F\}$ is nonempty. Let $G_1 = X$, and for each integer $n > 1$ let G_n be the set of all $(x_1, x_2, \dots, x_n) \in \Pi_{i=1}^n X$ such that $(x_{i+1}, x_i) \in F$ for $i = 1, \dots, n-1$. For each integer $n > 1$ and each $\alpha \in \Lambda$ let $G_{n,\alpha}$ be the set of all $(x_1, x_2, \dots, x_n) \in G_n$ such that $(x_2, x_1) \in F_\alpha$. Then, clearly each G_n is compact and $G_n = \bigcup_{\alpha\in\Lambda} G_{n,\alpha}$.

Note that G_2 is homeomorphic to F. So G_1 and G_2 are compact and connected. Assume $n > 2$ and G_{n-1} is connected. Let $\Psi_\alpha : G_{n,\alpha} \to G_{n-1}$ be the continuous function defined by $\Psi(\mathbf{x}) = \pi_{2,n}(\mathbf{x})$. If $\mathbf{y} =(y_1, y_2, \dots, y_{n-1}) \in G_{n-1}$, then $\Psi_\alpha^{-1}(\mathbf{y}) = \{(z, y_1, y_2, \dots, y_{n-1}) \mid (y_1, z) \in G_\alpha\}$ is homeomorphic to

$\{z \mid (y_1, z) \in G_\alpha\}$ which by assumption is nonempty and connected. Therefore Ψ_α is a monotone continuous surjection onto a compact connected set. It follows that $G_{n,\alpha}$ is connected for each α.

Note that since for each $y \in X$ the set $\{x \in X \mid (x, y) \in F\}$ is nonempty, each coordinate projection of G_n is X, and the projection onto the first two coordinates of G_n is F^{-1}. Now suppose H and K are nonempty closed subsets of G_n such that $G_n = H \cup K$. Let H^* be the set of all pairs $(a, b) \in F$ such that there is a $(y_1, y_2, \ldots y_n) \in H$ such that $b = x_1$ and $a = x_2$, and let K^* be the set of all pairs $(a, b) \in F$ such that there is a $(y_1, y_2, \ldots y_n) \in K$ such that $b = x_1$ and $a = x_2$. Since H^* and K^* are the respective projections of H and K onto their first two coordinates, H^* and K^* are continuous images of H and K, and therefore they are nonempty closed sets whose union is the connected set F. So $H^* \cap K^* \neq \emptyset$. Let $(c, d) \in H^* \cap K^*$. There exists $\mathbf{y} = (y_{1,} y_2, \ldots y_n) \in H$ such that $y_1 = c$ and $y_2 = d$, there exists $\mathbf{z} = (z_1, z_2, \ldots z_n) \in K$ such that $z_1 = c$ and $z_2 = d$, and there exist $\alpha \in \Lambda$ such that $(d, c) \in F_\alpha$. Thus, the connected set $G_{n,\alpha}$ contains both $\mathbf{y}$ and $\mathbf{z}$. It follows that $H \cap K \neq \emptyset$, and therefore G_n is connected.

By induction it follows that G_n is connected for each n. For each n let G_n^* be the set of all $(x_1, x_2, \ldots, x_n, \ldots) \in \Pi_{i=1}^{\infty} X$ such that $(x_1, x_2, \ldots x_n) \in G_n$. Then, G_n^* is compact and connected for each n, and since $\varprojlim F = \bigcap\limits_{n=1}^{\infty} G_n^*$, it follows that $\varprojlim F$ is connected . □

Lemma 2. *Suppose X is a compact Hausdorff continuum, and $f : X \to 2^X$ is an upper semi-continuous set valued function, and, for each n, G_n is the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^{n} X$ such that $x_{i+1} \in f(x_i)$ for $i = 1, \ldots, n-1$. Then $\varprojlim f$ is connected if and only if G_n is connected for each n .*

Proof. The proof is contained in the last two sentences of the proof of Theorem 1. □

Theorem 3. *Suppose X is a compact Hausdorff continuum, and $f : X \to 2^X$ is a surjective upper semi-continuous set valued function. Then $\varprojlim f$ is connected if and only if $\varprojlim f^{-1}$* is connected.

Proof. Assume X is a compact Hausdorff continuum, and $f : X \to 2^X$ is a surjective upper semi-continuous set valued function. For each n let G_n be the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^{n} X$ such that $x_i \in f(x_{i+1})$ for each i such that $1 \le i \le n-1$, and let G_n^{-1} be the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^{n} X$ such that $x_i \in f^{-1}(x_{i+1})$ for each i such that $1 \le i \le n-1$. Then $(x_1, x_2, \ldots, x_n) \in G_n$ if and only if $(x_n, x_{n-1}, \ldots, x_1) \in G_n^{-1}$. Therefore G_n and G_n^{-1} are homeomorphic. Since $\varprojlim f$ is connected if and only if G_n is connected for each n by Lemma 2, and $\varprojlim f^{-1}$

is connected if and only if G_n^{-1} is connected for each n, it follows that $\varprojlim f$ is connected if and only if $\varprojlim f^{-1}$ is connected. $\square$

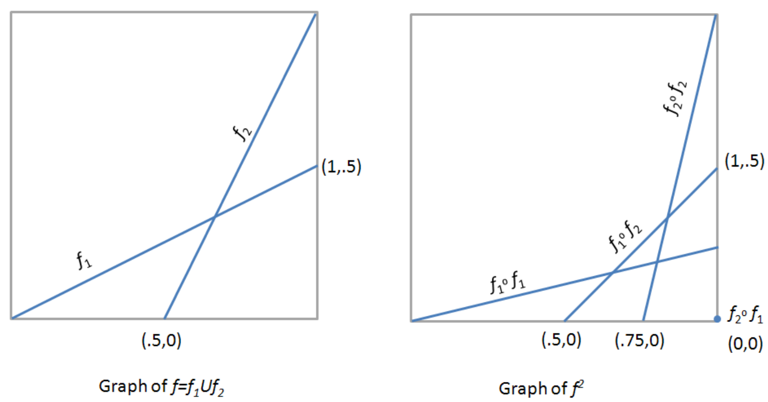

FIGURE 3.1.

Example 4. Define $f : [0,1] \to 2^{[0,1]}$ to be the function whose graph is the union of the following two sets: $A = \{(x,y) : 0 \leq x \leq 1 \text{ and } y = \frac{1}{2}x\}$ and $B = \{(x,y) : \frac{1}{2} \leq x \leq 1 \text{ and } y = 2x - 1\}$. In Figure 1, A is the graph of f_1, and B is the graph of f_2. The function f is upper semi-continuous since the graph of f is compact [2, Theorem 4, p. 58] , and the graph of f is clearly connected. It is easy to see that the graph of $f \circ f$ is not connected since the point $(1,0)$ is an isolated point in the graph of $f \circ f = f^2$. Therefore $\varprojlim f = \varprojlim (A \cup B)$ is not connected. Let us label $A_1 = \{(x,y) \in A : x \leq \frac{2}{3}\}$, $A_2 = \{(x,y) \in A : x \geq \frac{2}{3}\}$, $B_1 = \{(x,y) \in B : x \leq \frac{2}{3}\}$, and $B_2 = \{(x,y) \in B : x \geq \frac{2}{3}\}$. Then A and $A_1 \cup B_2$ are each the graph of a continuous function from $[0,1]$ into $[0,1]$. Also the set $A \cup (A_1 \cup B_2)$ is closed and connected and is the graph of a surjective upper semi-continuous function from $[0,1]$ to $2^{[0,1]}$. Therefore, by Theorem 1, $\varprojlim A \cup (A_1 \cup B_2) = \varprojlim A \cup B_2$ is connected, whereas it has been noted that $\varprojlim (A \cup B_2) \cup B_1 = \varprojlim A \cup B$ is not connected. Similarly, with the use of Theorems 1 and Theorem 3 it can be seen that $\varprojlim A_1 \cup B$ is connected but $\varprojlim (A_1 \cup B) \cup A_2 = \varprojlim A \cup B$ is not connected. This demonstrates the necessity in Theorem 1 for the assumption that each function have domain all of X. Also $A_1 \cup B$ is the graph of a very simple upper semi-continuous function with a connected inverse limit such that if one adds the set A which is the graph a straight line defined on all of $[0,1]$ one gets $A \cup B$ which has disconnected inverse limit. This raises the question that motivates the next two theorems. Which is if $\varprojlim f$ is connected then what sort of set can one add to the graph of f and obtain the graph of a set valued function with inverse limit that is still connected?

The following theorem was first suggested by Chris Mouron. Its usefulness is certainly hindered by the difficulty of checking the condition $fg = gf$ except of course in the case where g is the identity function.

Theorem 5. *Suppose X is a compact Hausdorff continuum, and $f : X \to 2^X$ is a surjective upper semi-continuous set valued function such that $\underleftarrow{\lim} f$ is connected, and $g : X \to X$ is a continuous function such that $fg = gf$, and the graphs of f and g are not disjoint. Then $\underleftarrow{\lim} f \cup g$ is connected.*

Proof. Assume X is a compact Hausdorff continuum, and $f : X \to 2^X$ is a surjective upper semi-continuous set valued function such that $\underleftarrow{\lim} f$ is connected, and $g : X \to X$ is a continuous function such that $fg = gf$, and the graphs of f and g are not disjoint. For each positive integer $n > 1$ let $G_n(f \cup g)$ be the set of all $(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} X$ such that $x_i \in f \cup g(x_{i+1})$ for $1 \leq i \leq n$, let $G_n(f)$ be the set of all $(x_1, x_2, \ldots, x_n) \in G_n(f \cup g)$ such that $x_i \in f(x_{i+1})$ for each $i \leq n$, and for each $j < n$ let $G_{n,j}$ be the set of all $(x_1, x_2, \ldots, x_n) \in G_n(f \cup g)$ such that $x_j = g(x_{j+1})$. We will show that $G_n(f \cup g)$ is connected for each $n > 1$. Since $G_2(f \cup g)$ is homeomorphic to the graph of $f \cup g$, it is connected. Assume $G_{n-1}(f \cup g)$ is connected.

From the definitions above it follows that $G_n(f \cup g) = G_n(f) \cup \bigcup_{j=1}^{n-1} G_{n,j}$. Since the graphs of f and g are not disjoint there is a point z in X such that $g(z) \in f(z)$, and for each $j < n$ there is an $\mathbf{x} \in G_n(f)$ such that $\pi_{j+1}(\mathbf{x}) = z$. Therefore $\mathbf{x} \in G_n(f) \cap G_{n,j}$. Since $\underleftarrow{\lim} f$ is connected, $G_n(f)$ is connected by Lemma ? So we will show that $G_{n,j}$ is connected for each $j < n$ from which it follows that $G_n(f \cup g)$ is connected.

To see that $G_{n,1}$ is connected note that the function that sends $(x_1, x_2, ..., x_n) \in G_{n-1}$ to $(g(x_1), x_1, x_2, ..., x_n) \in G_{n,1}$ is a homeomorphism from $G_{n-1}(f \cup g)$ onto $G_{n,1}$.

For each $j < n$ consider the function $\Psi_j : \Pi_{i=1}^{n} X \to \Pi_{i=1}^{n} X$ defined by $\Psi_i(\mathbf{x}) = \pi_{1,j}(\mathbf{x}) \oplus (g(\pi_{j+2}(\mathbf{x}))) \oplus \pi_{j+2,n}(\mathbf{x})$. It is obvious that each Ψ_j is continuous. We will show that the restriction of Ψ_j to $G_{n,j}$ maps $G_{n,j}$ onto $G_{n,j+1}$.

Let $\mathbf{x}$ be an element of $G_{n,j}$. That is, assume $\mathbf{x} \in G_n$, and assume $\pi_j(\mathbf{x}) = g(\pi_{j+1}(\mathbf{x}))$. Now either $\pi_{j+1}(\mathbf{x}) = g(\pi_{j+2}(\mathbf{x}))$ or $\pi_{j+1}(\mathbf{x}) \in f(\pi_{j+2}(\mathbf{x}))$. If $\pi_{j+1}(\mathbf{x}) = g(\pi_{j+2}(\mathbf{x}))$, then $\mathbf{x} \in G_{n,j+1}$, and $\Psi_j(\mathbf{x}) = \mathbf{x}$. So $\Psi_j(\mathbf{x})) \in G_{n,j+1}$. If $\pi_{j+1}(\mathbf{x}) \in f(\pi_{j+2}(\mathbf{x}))$ then $\pi_j(\mathbf{x}) \in g(f(\pi_{j+2}(\mathbf{x}))) = f(g(\pi_{j+2}(\mathbf{x}))$. So $\Psi_j(\mathbf{x}) = \pi_{1,j}(\mathbf{x}) \oplus (g(\pi_{j+2}(\mathbf{x}))) \oplus \pi_{j+2,n}(\mathbf{x})$ is an element of $G_{n,j+1}$. Therefore Ψ_j maps $G_{n,j}$ into $G_{n,j+1}$.

Now let $\mathbf{x}$ be an element of $G_{n,j+1}$. That is, assume $\mathbf{x} \in G_n$, and assume $\pi_{j+1}(\mathbf{x}) = g(\pi_{j+2}(\mathbf{x})$. Now either $\pi_j(\mathbf{x}) = g(\pi_{j+1}(\mathbf{x}))$ or $\pi_j(\mathbf{x}) \in f(\pi_{j+1}(\mathbf{x}))$. If $\pi_j(\mathbf{x}) = g(\pi_{j+1}(\mathbf{x}))$, then $\mathbf{x} \in G_{n,j}$, and $\Psi_j(\mathbf{x}) = \mathbf{x}$. So $\mathbf{x} \in \Psi_j(G_{n,j})$. If

$\pi_j(\mathbf{x}) \in f(\pi_{j+1}(\mathbf{x}))$ then $\pi_j(\mathbf{x}) \in f(g(\pi_{j+2}(\mathbf{x}))) = g(f(\pi_{j+2}(\mathbf{x})))$. So there is a $z \in f(\pi_{j+2}(\mathbf{x}))$ such that $\pi_j(\mathbf{x}) = g(z)$. Therefore $\mathbf{w} = \pi_{1,j}(\mathbf{x}) \oplus (z) \oplus \pi_{j+2,n}(\mathbf{x})$ is an element of $G_{n,j}$, and $\Psi_j(\mathbf{w}) = \mathbf{x}$. Again this implies that $\mathbf{x} \in \Psi_j(G_{n,j})$. Therefore Ψ_j maps $G_{n,j}$ onto $G_{n,j+1}$.

It follows then that each $G_{n,j}$ is connected, and therefore, G_n is connected. By induction we have that each G_n is connected. So, from Lemma 2 it follows that $\varprojlim f \cup g$ is connected. □

The example above shows that one must be very careful about what one adds to the graph of a function whose inverse limit is connected in order to have the union of the two graphs be a function with connected inverse limit. For example it is possible to add the graph of a straight line defined on all of $[0,1]$ to the graph of a very simple set valued function $f : [0,1] \to [0,1]$ with connected $\varprojlim f$ and have the inverse limit be not connected. We will show that under some conditions one can add a section of the graph of the identity function or a section of the graph of a constant function and the inverse limit will remain connected.

Theorem 6. *Suppose X is a compact Hausdorff continuum, and $f : X \to 2^X$ is a surjective upper semi-continuous set valued function such that $\varprojlim f$ is connected, D is a closed subset of X, and $g : D \to X$ is a function such that the graph of $f \cup g$ is connected, and if x is in the boundary of D in X, then $g(x) \in f(x)$. If, in addition, the function g is defined by $g(x) = x$ for each $x \in D$ or for some $a \in X$ the function g is defined by $g(x) = a$ for each $x \in D$, then $\varprojlim f \cup g$ is connected.*

Proof. Assume X is a compact Hausdorff continuum, and $f : X \to 2^X$ is a surjective upper semi-continuous set valued function such that $\varprojlim f$ is connected, D is a closed subset of X, and $g : D \to X$ is a function such that the graph of $f \cup g$ is connected, and if x is in the boundary of D in X, then $g(x) \in f(x)$. Assume also that $g(x) = x$ for each $x \in D$.

For each natural number n let $G_n(f \cup g)$ be the set of all $(x_1, x_2, ..., x_n) \in \Pi_{i=1}^n X$ such that $x_i \in f \cup g(x_{i+1})$ for $1 \leq i \leq n$, let $G_n(f)$ be the set of all $(x_1, x_2, ..., x_n) \in \Pi_{i=1}^n X$ such that $x_i \in f(x_{i+1})$ for $1 \leq i \leq n$, and for each $0 \leq j \leq n-1$ let G_n^j be the set of all $(x_1, x_2, ..., x_n) \in G_n(f \cup g)$ such that $x_i \in f(x_{i+1})$ for $n - j \leq i \leq n$. Note that for each n we have $G_n(f) = G_n^{n-1} \subset G_n^{n-2} \subset \cdots \subset G_n^0 = G_n(f \cup g)$, and note that $G_n(f)$ is connected for each n since $\varprojlim f$ is connected.

Note also that $G_2^1 = G_2(f)$, which is connected, and G_2^0 is homeomorphic to the graph of $f \cup g$ which is connected.

Next we will show that if $n > 2$ and G_{n-1}^j is connected for each $1 \leq j \leq n-2$, then G_n^j is connected for each $1 \leq k \leq n-1$. So assume $n > 2$ and G_{n-1}^j is connected for each $1 \leq j \leq n-2$. Let k be a non negative integer such that $k < n - 1$. Let $\mathbf{x} = (x_1, x_2, ..., x_n) \in G_n^k$. It will be shown that either $\mathbf{x} \in G_n^{k+1}$ or there is a continuum in G_n^k that contains $\mathbf{x}$ and a point in G_n^{k+1}. So, assume $\mathbf{x}$ is not in G_n^{k+1}. Then $\pi_{n-k-1}(\mathbf{x}) \in f \cup g(\pi_{n-k}(\mathbf{x}))$, and $\pi_{n-k-1}(\mathbf{x}) \in X \setminus f(\pi_{n-k}(\mathbf{x}))$. So, $\pi_{n-k-1}(\mathbf{x}) = g(\pi_{n-k}(\mathbf{x})) = \pi_{n-k}(\mathbf{x})$, and $\pi_{n-k}(\mathbf{x}) \in D$. Let $\mathbf{x}' = \pi_{1,n-k-2}(\mathbf{x}) \oplus \pi_{n-k,n}(\mathbf{x})$. That is, $\mathbf{x}'$ is obtained by removing the $(n-k-1)^{th}$ coordinate of $\mathbf{x}$. Note that $\mathbf{x}' \in G_{n-1}^k$.

Since the graphs of f and g are closed and the graph of $f \cup g$ is connected, there is a point $\mathbf{y}$ in G_{n-1}^k such that $\pi_{n-k}(\mathbf{y}) \in D$, and $\pi_{n-k}(\mathbf{y}) = g(\pi_{n-k}(\mathbf{y})) \in f(\pi_{n-k}(\mathbf{y}))$. Since G_{n-1}^k is connected, there is a continuum J in G_{n-1}^k that contains $\mathbf{x}'$ and $\mathbf{y}$. If $D \subset \pi_{n-k}(J)$, then let $K = J$, and let $\mathbf{y}' = \mathbf{y}$. If D is not contained in $\pi_{n-k}(J)$, then let W be the set of all $\mathbf{z} \in J$ such that $\pi_{n-k}(\mathbf{z}) \in D$, and let K be the component of W which contains $\mathbf{x}'$. Then K contains a point $\mathbf{y}'$ in the boundary of W in J. It follows that $\pi_{n-k}(\mathbf{y}')$ is in the boundary of D in X, and therefore $\pi_{n-k}(\mathbf{y}') = g(\pi_{n-k}(\mathbf{y}')) \in f(\pi_{n-k}(\mathbf{y}'))$. In either case K is a continuum such that $\pi_{n-k}(K) \subset D$ and K contains $\mathbf{x}'$ and a point $\mathbf{y}'$ such that $\pi_{n-k}(\mathbf{y}') = g(\pi_{n-k}(\mathbf{y}')) \in f(\pi_{n-k}(\mathbf{y}'))$. Now let $F : K \to G_n^k$ be defined by $F(\mathbf{z}) = \pi_{1,n-k}(\mathbf{z}) \oplus \pi_{n-k,n-1}(\mathbf{z})$. That is, insert a new coordinate between the $(k-1)^{th}$ coordinate and the k^{th} coordinate of $\mathbf{z}$ equal to the k^{th} coordinate of $\mathbf{z}$. This map F is clearly a homeomorphism on K, and $K^* = F(K)$ is a continuum in G_n^k that contains $\mathbf{x}$ since $\mathbf{x} = F(\mathbf{x}')$ and the point $F(\mathbf{y}')$ which is in G_n^{k+1} .

By the same argument either $F(\mathbf{y}')$ is contained in G_n^{k+1} or there is a continuum in G_n^{k+1} that contains $F(\mathbf{y}')$ and a point in G_n^{k+2}. Continuing in this way there is a continuum in G_n^k that contains $\mathbf{x}$ and a point in $G_n^{n-1} = G_n(f)$. Since $G_n(f) = G_n^{n-1} \subset G_n^{n-2} \subset \cdots \subset G_n^k$, and $G_n(f)$ is connected, it follows that G_n^k is connected.

This concludes the inductive proof that G_n^j is connected for each n and each $j \leq n-1$. But then G_n^0 is connected for each n, and $G_n^0 = G_n(f \cup g)$. It follows that $\varprojlim f \cup g$ is connected.

Now assume there is an $a \in X$ such that $g(x) = a$ for each $x \in D$. The proof for this case begins just like the proof above.

Note that $G_2^1 = G_2(f)$, which must be connected since $\varprojlim f$ is connected, and G_2^0 is homeomorphic to the graph of $f \cup g$ which is connected.

Next we will show that if $n > 2$ and G_{n-1}^j is connected for each $0 < j \leq n-2$, then G_n^j is connected for each $1 \leq k \leq n-1$. So assume $n > 2$ and G_{n-1}^j is connected for each non negative integer $j \leq n-2$. Let k be a non negative integer such that $k < n-1$. Let $\mathbf{x} = (x_1, x_2, ..., x_n) \in G_n^k$. It will be shown that either $\mathbf{x} \in G_n^{k+1}$ or there is a continuum in G_n^k that contains $\mathbf{x}$ and a point in G_n^{k+1}. So, assume $\mathbf{x}$ is not in G_n^{k+1}. Then $\pi_{n-k-1}(\mathbf{x}) \in f \cup g(\pi_{n-k}(\mathbf{x}))$, and $\pi_{n-k-1}(\mathbf{x}) \in X \setminus f(\pi_{n-k}(\mathbf{x}))$. Up to this point the proof has been identical to the proof above for $g(x) = x$. Here there is a small difference. In this case what follows is that $\pi_{n-k-1}(\mathbf{x}) = a = g(\pi_{n-k}(\mathbf{x}))$ and $\pi_{n-k}(\mathbf{x}) \in D$. So let $\mathbf{x}' = \pi_{n-k,n}(\mathbf{x})$ and note that in this case $\mathbf{x}' \in G^k_{n-k+1} = G_{n-k+1}(f)$.

Since the graphs of f and g are closed, and the graph of $f \cup g$ is connected, there is a point $\mathbf{y}$ in G^k_{n-k+1} such that $\pi_1(\mathbf{y}) \in D$, and $\pi_1(\mathbf{y}) = g(\pi_1(\mathbf{y})) \in f(\pi_1(\mathbf{y}))$. Since G^k_{n-k+1} is connected, there is a continuum J in G^k_{n-k+1} that contains $\mathbf{x}'$ and $\mathbf{y}$. If $D \subset \pi_1(J)$, then let $K = J$, and let $\mathbf{y}' = \mathbf{y}$. If D is not contained in $\pi_1(J)$, then let W be the set of all $\mathbf{z} \in J$ such that $\pi_1(\mathbf{z}) \in D$, and let K be the component of W which contains $\mathbf{x}'$. Then K contains a point $\mathbf{y}'$ in the boundary of W in J. It follows that $\pi_1(\mathbf{y}')$ is in the boundary of D in X, and therefore $\pi_1(\mathbf{y}') = g(\pi_1(\mathbf{y}')) \in f(\pi_1(\mathbf{y}'))$. In either case K is a continuum such that $\pi_{n-k}(K) \subset D$, and K contains $\mathbf{x}'$ and a point $\mathbf{y}'$ such that $\pi_1(\mathbf{y}') = g(\pi_1(\mathbf{y}')) \in f(\pi_1(\mathbf{y}'))$. Note that since the first coordinate of each point K is in D if we attach $\pi_{1,n-k-1}(\mathbf{x})$ to any point in K the result is a point G^k_n. That is, let $F : K \to G^k_n$ be defined by $F(\mathbf{z}) = \pi_{1,n-k-1}(\mathbf{x}) \oplus \mathbf{z}$. This map F is clearly a homeomorphism on K, and $K^* = F(K)$ is a continuum in G^k_n that contains $\mathbf{x}$ since $\mathbf{x} = F(\mathbf{x}')$, and the point $F(\mathbf{y}')$ which is in G^{k+1}_n . (The rest of the proof is identical to the case where $g(x) = x$. It is included below for completeness.)

By the same argument either $F(\mathbf{y}')$ is contained in G^{k+1}_n or there is a continuum in G^{k+1}_n that contains $F(\mathbf{y}')$ and a point in G^{k+2}_n. Continuing in this way there is a continuum in G^k_n that contains $\mathbf{x}$ and a point in $G^{n-1}_n = G_n(f)$. Since $G_n(f) = G^{n-1}_n \subset G^{n-2}_n \subset \cdots \subset G^k_n$, and $G_n(f)$ is connected, it follows that G^k_n is connected.

This concludes the inductive proof that G^j_n is connected for each n and each $j \leq n-1$. But then G^0_n is connected for each n, and $G^0_n = G_n(f \cup g)$. It follows that $\varprojlim f \cup g$ is connected. $\square$

When we apply the results in Theorem 6 and Theorem 3 to the case where $f : [0,1] \to 2^{[0,1]}$ and $\varprojlim f$ is connected we see that if we add to the graph of f a horizontal line of the form $\{(x,a) : c \leq x \leq d\}$ where $\{c,d\} \subset f^{-1}(a) \cup \{0,1\}$ or we add to the graph of f a vertical line of the form $\{(a,x) : c \leq x \leq d\}$ where $\{c,d\} \subset f(a) \cup \{0,1\}$ then the inverse limit with this new set valued function will be connected.

For an upper semi-continuous set valued function $f : X \to 2^X$, and a continuous function $g : X \to X$, the upper semi-continuous set valued function $g^{-1}fg$ is given by $y \in g^{-1}fg(x)$ if and only if $g(y) \in f(g(x))$. We say an upper semi-continuous function $h : X \to 2^X$ is a semi-conjugate of an upper semi-continuous function $f : X \to 2^X$ if and only if there is a continuous surjective function $g : X \to X$ such that $gh = fg$. It is easy to check that this requirement is equivalent to saying $h = g^{-1}fg$. It is also easy to see that h being semi-conjugate of f does not imply that f is a semi-conjugate of h.

Theorem 7. *Suppose X is a compact Hausdorff continuum, $f : X \to 2^X$ is a surjective upper semi-continuous set valued function, $g : X \to X$ is continuous and surjective, and $\varprojlim g^{-1}fg$ is connected, then $\varprojlim f$ is connected.*

Proof. Assume X is a compact Hausdorff continuum, $f : X \to 2^X$ is a surjective upper semi-continuous set valued function, $g : X \to X$ is continuous and surjective, and $\varprojlim g^{-1}fg$ is connected. For each n let G_n be the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^n X$ such that $x_i \in f(x_{i+1})$ for $i \leq n-1$, and for each n let G'_n be the set of all $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^n X$ such that $x_i \in g^{-1}f(g(x_{i+1}))$ for $i \leq n-1$. It will be shown that the continuous function that sends $(x_1, x_2, ..., x_n)$ to $(g(x_1), g(x_2), \ldots, g(x_n))$ maps G'_n onto G_n.

Let $(x_1, x_2, \ldots, x_n)$ be an element of G'_n. Since $x_i \in g^{-1}fg(x_{i+1})$ for each $i \leq n-1$, it is true that $g(x_i) \in f(g(x_{i+1}))$ for each $i \leq n-1$. Therefore $(g(x_1), g(x_2), \ldots, g(x_n)) \in G_n$. Now for each $(y_1, y_2, \ldots, y_n) \in G_n$, let $(x_1, x_2, ..., x_n)$ be an element of $\Pi_{i=1}^n X$ such $x_i \in g^{-1}(y_i)$ for each $i \leq n$. Since for each $i \leq n$ it is true that $y_i \in f(y_{i+1}) = f(g(x_{i+1}))$, it follows that for each $i \leq n$ it is true that $x_i \in g^{-1}(y_i) \subset g^{-1}f(g(x_{i+1}))$. Thus $(x_1, x_2, ..., x_n) \in G'_n$. Therefore the continuous function that sends $(x_1, x_2, ..., x_n)$ to $(g(x_1), g(x_2), \ldots, g(x_n))$ maps G'_n onto G_n.

Since $\varprojlim g^{-1}fg$ is connected, G'_n is connected for each n. Therefore G_n is connected for each n. Thus $\varprojlim f$ is connected by Lemma 2. □

The previous theorem is most likely to be useful for producing new functions with disconnected inverse limit since if $f : X \to 2^X$ is a set valued function such that $\varprojlim f$ is not connected, then for any continuous function $g : X \to X$ the $\varprojlim g^{-1}fg$ will also be not connected.

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