CHAPTER 12
SORTING AND SELECTION

ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN JAVA, GOODRICH, TAMASSIA AND GOLDWASSER (WILEY 2016)
DIVIDE AND CONQUER ALGORITHMS
DIVIDE AND CONQUER ALGORITHMS
ANALYSIS WITH RECURRENCE EQUATIONS

- Divide-and-conquer is a general algorithm design paradigm:
  - Divide: divide the input data $S$ into $k$ (disjoint) subsets $S_1, S_2, ..., S_k$
  - Recur: solve the subproblems recursively
  - Conquer: combine the solutions for $S_1, S_2, ..., S_k$ into a solution for $S$

- The base case for the recursion are subproblems of constant size

- Analysis can be done using recurrence equations (relations)
• When the size of all subproblems is the same (frequently the case) the recurrence equation representing the algorithm is:

\[ T(n) = D(n) + kT\left(\frac{n}{c}\right) + C(n) \]

• Where
  • \( D(n) \) is the cost of dividing \( S \) into the \( k \) subproblems \( S_1, S_2, ..., S_k \)
  • There are \( k \) subproblems, each of size \( \frac{n}{c} \) that will be solved recursively
  • \( C(n) \) is the cost of combining the subproblem solutions to get the solution for \( S \)
**EXERCISE**

**RECURSIVE EQUATION SETUP**

- Algorithm – transform multiplication of two \( n \)-bit integers \( I \) and \( J \) into multiplication of \( \left(\frac{n}{2}\right) \)-bit integers and some additions/shifts

1. Where does recursion happen in this algorithm?

2. Rewrite the step(s) of the algorithm to show this clearly.

---

**Algorithm** \texttt{multiply}(I,J)

\textbf{Input:} \( n \)-bit integers \( I,J \)

\textbf{Output:} \( I \times J \)

1. if \( n > 1 \) then

2. Split \( I \) and \( J \) into high and low order halves:
   \( I_h, I_l, J_h, J_l \)

3. \( x_1 \leftarrow I_h \times J_h; \ x_2 \leftarrow I_h \times J_l; \)

4. \( x_3 \leftarrow I_l \times J_h; \ x_4 \leftarrow I_l \times J_l \)

5. \( Z \leftarrow x_1 \times 2^n + x_2 \times 2^{n \over 2} + x_3 \times 2^{n \over 2} + x_4 \)

6. else

7. \( Z \leftarrow I \times J \)

8. return \( Z \)
EXERCISE
RECURRENCE EQUATION SETUP

• Algorithm – transform multiplication of two 
n-bit integers \( I \) and \( J \) into multiplication of 
\( \left( \frac{n}{2} \right) \)-bit integers and some additions/shifts

3. Assuming that additions and shifts of \( n \)-bit
numbers can be done in \( O(n) \) time,
describe a recurrence equation showing
the running time of this multiplication
algorithm

**Algorithm** multiply\((I, J)\)

**Input:** \( n \)-bit integers \( I, J \)

**Output:** \( I \ast J \)

1. if \( n > 1 \) then
2. Split \( I \) and \( J \) into high 
and low order halves:
\( I_h, I_l, J_h, J_l \)
3. \( x_1 \leftarrow \text{multiply}(I_h, J_h); \ x_2 \leftarrow \text{multiply}(I_h, J_l) \)
4. \( x_3 \leftarrow \text{multiply}(I_l, J_h); \ x_4 \leftarrow \text{multiply}(I_l, J_l) \)
5. \( Z \leftarrow x_1 \ast 2^n + x_2 \ast 2^n + x_3 \ast 2^n + x_4 \)
6. else
7. \( Z \leftarrow I \ast J \)
8. return \( Z \)
EXERCISE

RECURRANCE EQUATION SETUP

• Algorithm – transform multiplication of two $n$-bit integers $I$ and $J$ into multiplication of $\left(\frac{n}{2}\right)$-bit integers and some additions/shifts

• The recurrence equation for this algorithm is:
  • $T(n) = 4T\left(\frac{n}{2}\right) + O(n)$
  • The solution is $O(n^2)$ which is the same as naïve algorithm

Algorithm multiply($I,J$)

Input: $n$-bit integers $I,J$
Output: $I*J$

1. if $n > 1$ then
2. Split $I$ and $J$ into high and low order halves:
   $I_h, I_l, J_h, J_l$
3. $x_1 \leftarrow$ multiply($I_h, J_h$); $x_2 \leftarrow$ multiply($I_h, J_l$)
4. $x_3 \leftarrow$ multiply($I_l, J_h$); $x_4 \leftarrow$ multiply($I_l, J_l$)
5. $Z \leftarrow x_1 * 2^n + x_2 * 2^{\frac{n}{2}} + x_3 * 2^{\frac{n}{2}} + x_4$
6. else
7. $Z \leftarrow I*J$
8. return $Z$
• Remaining question: how do we solve recurrence relations?
  • **Iterative substitution** — continually expand a recurrence to yield a summation, then bound the summation
  • **Analyze the recursion tree** — determine work per level and number of levels in a recursion tree. This is not a proof technique, more of an intuitive sketch of a proof
  • **Master theorem (method)** — rule to go directly to solution of recurrence. This is slightly beyond scope of course, but we will see it anyway
ITERATIVE SUBSTITUTION

- In the iterative substitution, or “plug-and-chug,” technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern. Example:
  - \( T(n) = 2T\left(\frac{n}{2}\right) + bn \)
  - \( = 2\left(2T\left(\frac{n}{2}\right) + b\left(\frac{n}{2}\right)\right) + bn = 2^2T\left(\frac{n}{2^2}\right) + 2bn \)
  - \( = 2^3T\left(\frac{n}{2^3}\right) + 3bn \)
  - \( = \ldots \)
  - \( = 2^iT\left(\frac{n}{2^i}\right) + ibn \)
  - Note that base, \( T(n) = b, \) case occurs when \( 2^i = n. \) That is, \( i = \log n. \)
  - So,
    \[ T(n) = bn + n\log n = O(n\log n) \]
THE RECURSION TREE

• Draw the recursion tree for the recurrence relation and look for a pattern.

Example: \( T(n) = 2T\left(\frac{n}{2}\right) + bn \)

<table>
<thead>
<tr>
<th>depth</th>
<th>T’s size</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 ( n )</td>
<td>( bn )</td>
</tr>
<tr>
<td>1</td>
<td>2 ( n/2 )</td>
<td>( bn )</td>
</tr>
<tr>
<td>( i )</td>
<td>( 2^i n/2^i )</td>
<td>( bn )</td>
</tr>
<tr>
<td>( ... )</td>
<td>( ... )</td>
<td>( ... )</td>
</tr>
</tbody>
</table>

• Total time: \( bn + bn \log n = O(n \log n) \)
THE MASTER THEOREM (METHOD)

• Many divide-and-conquer algorithms have the form:

\[ T(n) = aT \left( \frac{n}{b} \right) + f(n) \]

• The master theorem:

1. If \( f(n) \) is \( O \left( n^{\log_b a - \epsilon} \right) \), then \( T(n) \) is \( \Theta \left( n^{\log_b a} \right) \)
2. If \( f(n) \) is \( \Theta \left( n^{\log_b a \log^k n} \right) \), then \( T(n) \) is \( \Theta \left( n^{\log_b a \log^{k+1} n} \right) \)
3. If \( f(n) \) is \( \Omega \left( n^{\log_b a + \epsilon} \right) \), then \( T(n) \) is \( \Theta(f(n)) \), provided \( af \left( \frac{n}{b} \right) \leq \delta f(n) \) for some \( \delta < 1 \)

• Examples

- \( T(n) = 4T \left( \frac{n}{2} \right) + n \)
  - \( O(n^2) \)
- \( T(n) = T \left( \frac{n}{2} \right) + 1 \)
  - \( O(\log n) \), (binary search)
- \( T(n) = T \left( \frac{n}{3} \right) + n \log n \)
  - \( O(n \log n) \)
MERGE SORT

7 2 9 4 → 2 4 7 9

7 2 → 2 7
9 4 → 4 9

7 → 7 2 → 2
9 → 9 4 → 4
Merge-Sort

- Merge-sort is based on the divide-and-conquer paradigm. It consists of three steps:
  - Divide: partition input sequence $S$ into two sequences $S_1$ and $S_2$ of about $\frac{n}{2}$ elements each
  - Recur: recursively sort $S_1$ and $S_2$
  - Conquer: merge $S_1$ and $S_2$ into a sorted sequence

- What is the recurrence relation?

Algorithm `mergeSort(S, C)`

Input: Sequence $S$ of $n$ elements, Comparator $C$

Output: Sequence $S$ sorted according to $C$

1. if $S$.size() > 1 then
2. $(S_1, S_2) \leftarrow \text{partition}(S, \frac{n}{2})$
3. $S_1 \leftarrow \text{mergeSort}(S_1, C)$
4. $S_2 \leftarrow \text{mergeSort}(S_2, C)$
5. $S \leftarrow \text{merge}(S_1, S_2)$
6. return $S$
The running time of Merge Sort can be expressed by the recurrence equation:

\[ T(n) = 2T\left(\frac{n}{2}\right) + M(n) \]

We need to determine \( M(n) \), the time to merge two sorted sequences each of size \( \frac{n}{2} \).

**Algorithm mergeSort(\( S, C \))**

**Input:** Sequence \( S \) of \( n \) elements, Comparator \( C \)

**Output:** Sequence \( S \) sorted according to \( C \)

1. if \( S.\text{size()} > 1 \) then
2. \( (S_1, S_2) \leftarrow \text{partition}(S, \frac{n}{2}) \)
3. \( S_1 \leftarrow \text{mergeSort}(S_1, C) \)
4. \( S_2 \leftarrow \text{mergeSort}(S_2, C) \)
5. \( S \leftarrow \text{merge}(S_1, S_2) \)
6. return \( S \)
MERGING TWO SORTED SEQUENCES

• The conquer step of merge-sort consists of merging two sorted sequences \( A \) and \( B \) into a sorted sequence \( S \) containing the union of the elements of \( A \) and \( B \)

• Merging two sorted sequences, each with \( \frac{n}{2} \) elements and implemented by means of a doubly linked list, takes \( O(n) \) time
  • \( M(n) = O(n) \)

Algorithm \( \text{merge}(A,B) \)

Input: Sequences \( A,B \) with \( \frac{n}{2} \) elements each
Output: Sorted sequence of \( A \cup B \)

1. \( S \leftarrow \emptyset \)
2. while \( \neg A.\text{isEmpty()} \land \neg B.\text{isEmpty()} \) do
3.   if \( A.\text{first()} < B.\text{first()} \) then
4.     \( S.\text{addLast}(A.\text{removeFirst}()) \)
5.   else
6.     \( S.\text{addLast}(B.\text{removeFirst}()) \)
7. while \( \neg A.\text{isEmpty()} \) do
8. \( S.\text{addLast}(A.\text{removeFirst}()) \)
9. while \( \neg B.\text{isEmpty()} \) do
10. \( S.\text{addLast}(B.\text{removeFirst}()) \)
11. return \( S \)
• So, the running time of Merge Sort can be expressed by the recurrence equation:

\[
T(n) = 2T\left(\frac{n}{2}\right) + M(n)
\]

\[
= 2T\left(\frac{n}{2}\right) + O(n)
\]

\[
= O(n \log n)
\]

**Algorithm**

`mergeSort(S, C)`

**Input:** Sequence `S` of `n` elements, Comparator `C`

**Output:** Sequence `S` sorted according to `C`

1. if `S.size() > 1` then

2. `(S_1, S_2) ← partition(S, \frac{n}{2})`

3. `S_1 ← mergeSort(S_1, C)`

4. `S_2 ← mergeSort(S_2, C)`

5. `S ← merge(S_1, S_2)`

6. return `S`
An execution of merge-sort is depicted by a binary tree:

- Each node represents a recursive call of merge-sort and stores:
  - Unsorted sequence before the execution and its partition
  - Sorted sequence at the end of the execution
- The root is the initial call
- The leaves are calls on subsequences of size 0 or 1
EXECUTION EXAMPLE

• Partition

```
7  2  9  4
6  3  8  1

7  2
9  4
3  8
6  1
9  4
3  8
6  1
1  6
2  7
2  7
9  4
3  8
6  1
8  6
1  6
2  7
2  7
9  4
3  8
6  1
8  6
1  6

7  2  9  4 | 3  8  6  1
```

```
```

```

```

```

```

```
EXECUTION EXAMPLE

- Recursive Call, partition

```
7 2 9 4 | 3 8 6 1
```

```
7 2 | 9 4
```

```
7 | 2 9 4
```

```
3 8 6 1
```
EXECUTION EXAMPLE

- Recursive Call, partition
EXECUTION EXAMPLE

• Recursive Call, base case
EXECUTION EXAMPLE

• Recursive Call, base case
EXECUTION EXAMPLE

- Merge
EXECUTION EXAMPLE

- Recursive call, ..., base case, merge
EXECUTION EXAMPLE

- **Merge**

```
7 2 9 4 | 3 8 6 1
```

```
7 2 | 9 4 → 2 4 7 9
```

```
7 | 2 → 2 7
```

```
9 | 4 → 4 9
```

```
7 → 7 2 → 2 9 → 9 4 → 4
```

```
3 8 6 1
```

```
1 6
```

```
2 7 9
```

```
4 9
```

```
3 8 6 1
```

```
1 6
```

```
2 9
```

```
4 3
```

```
8 6
```

```
1 6
```

```
2 7 9
```

```
4 3
```

```
8 6
```

```
1 6
```

```
2 7 9
```

```
4 3
```

```
8 6
```

```
1 6
```

```
2 7 9
```

```
4 3
```

```
8 6
```

```
1 6
```
EXECUTION EXAMPLE

- Recursive call, ..., merge, merge
EXECUTION EXAMPLE

• Merge

```
| 7 2 9 4 | 3 8 6 1 | 1 2 3 4 6 7 8 9 |
```

```
| 7 2 | 9 4 | 2 4 7 9 |
| 7 2 | 2 7 |
| 7 | 7 |
```

```
| 3 8 | 6 1 | 1 3 8 6 |
| 3 8 | 3 8 |
| 3 8 |
```

```
7 2 9 4 3 8 6 1 1 2 3 4 6 7 8 9
```
ANOTHER ANALYSIS OF MERGE-SORT

- The height $h$ of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide in half the sequence,

- The work done at each level is $O(n)$
  - At level $i$, we partition and merge $2^i$ sequences of size $\frac{n}{2^i}$

- Thus, the total running time of merge-sort is $O(n \log n)$

<table>
<thead>
<tr>
<th>depth</th>
<th>#seqs</th>
<th>size</th>
<th>Cost for level</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$n$</td>
<td>$n$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$n/2$</td>
<td>$n$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$i$</td>
<td>$2^i$</td>
<td>$\frac{n}{2^i}$</td>
<td>$n$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$\log n$</td>
<td>$2^{\log n} = n$</td>
<td>$\frac{n}{2^{\log n}} = 1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>
## Summary of Sorting Algorithms (So Far)

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Time</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selection Sort</td>
<td>$O(n^2)$</td>
<td>Slow, in-place For small data sets (&lt; 1K)</td>
</tr>
<tr>
<td>Insertion Sort</td>
<td>$O(n^2)$ WC, AC</td>
<td>Slow, in-place For small data sets (&lt; 1K)</td>
</tr>
<tr>
<td></td>
<td>$O(n)$ BC</td>
<td></td>
</tr>
<tr>
<td>Heap Sort</td>
<td>$O(n \log n)$</td>
<td>Fast, in-place For large data sets (1K – 1M)</td>
</tr>
<tr>
<td>Merge Sort</td>
<td>$O(n \log n)$</td>
<td>Fast, sequential data access For huge data sets (&gt;1M)</td>
</tr>
</tbody>
</table>
QUICK-SORT
QUICK-SORT

• **Quick-sort** is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  • Divide: pick a random element \( x \) (called **pivot**) and partition \( S \) into
    • \( L \) - elements less than \( x \)
    • \( E \) - elements equal \( x \)
    • \( G \) - elements greater than \( x \)
  • Recur: sort \( L \) and \( G \)
  • Conquer: join \( L \), \( E \), and \( G \)
ANALYSIS OF QUICK SORT USING RECURRENCE RELATIONS

• Assumption: random pivot expected to give equal sized sublists

• The running time of Quick Sort can be expressed as:
  \[ T(n) = 2T\left(\frac{n}{2}\right) + P(n) \]

• \( P(n) \) - time to partition on input of size \( n \)

Algorithm quickSort(S, l, r)

Input: Sequence \( S \), indices \( l \), \( r \)

Output: Sequence \( S \) with the elements between \( l \) and \( r \) sorted

1. if \( l \geq r \) then
2. return \( S \)
3. \( i \leftarrow \text{rand()}\% (r - l) + l \) //random integer
4. \( h, k \leftarrow \text{partition}(x) \) //between \( l \) and \( r \)
5. \( x \leftarrow S\text{.at}(i) \)
6. quickSort(S, \( l, h - 1 \))
7. quickSort(S, \( k + 1, r \))
9. return \( S \)
PARTITION

• We partition an input sequence as follows:
  • We remove, in turn, each element \( y \) from \( S \) and
  • We insert \( y \) into \( L, E, \) or \( G, \) depending on the result of the comparison with the pivot \( x \)

• Each insertion and removal is at the beginning or at the end of a sequence, and hence takes \( O(1) \) time

• Thus, the partition step of quick-sort takes \( O(n) \) time

**Algorithm partition** \((S, p)\)

**Input:** Sequence \( S \), position \( p \) of the pivot

**Output:** Subsequences \( L, E, G \) of the elements of \( S \) less than, equal to, or greater than the pivot, respectively

1. \( L, E, G \leftarrow \emptyset \)
2. \( x \leftarrow S.remove(p) \)
3. while \( \neg S.isEmpty() \) do
4. \( y \leftarrow S.removeFirst() \)
5. if \( y < x \) then
6. \( L.addLast(y) \)
7. else if \( y = x \) then
8. \( E.addLast(y) \)
9. else // \( y > x \)
10. \( G.addLast(y) \)
11. return \( L, E, G \)
SO, THE EXPECTED COMPLEXITY OF QUICK SORT

- Assumption: random pivot expected to give equal sized sublists
- The running time of Quick Sort can be expressed as:
  \[ T(n) = 2T\left(\frac{n}{2}\right) + P(n) \]
  \[ = 2T\left(\frac{n}{2}\right) + O(n) \]
  \[ = O(n\log n) \]

Algorithm \texttt{quickSort}(S,l,r)
\textbf{Input:} Sequence $S$, indices $l$, $r$
\textbf{Output:} Sequence $S$ with the elements between $l$ and $r$ sorted

1. if $l \geq r$ then
2. \hspace{1em} return $S$
3. $i \leftarrow \text{rand()} \%(r - l) + l$ //random integer
4. \hspace{1em} //between $l$ and $r$
5. $x \leftarrow S.at(i)$
6. $(h,k) \leftarrow \text{partition}(x)$
7. quickSort($S$, $l$, $h - 1$)
8. quickSort($S$, $k + 1$, $r$)
9. return $S$
A QUICK-SORT TREE

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1

```
<table>
<thead>
<tr>
<th>7 4 9 6 2</th>
<th>2 4 6 7 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 2   → 2 4</td>
<td>7 9   → 7 9</td>
</tr>
<tr>
<td>2 → 2</td>
<td>9 → 9</td>
</tr>
</tbody>
</table>
```
EXECUTION EXAMPLE

- Pivot selection

```
7 2 9 4 3 7 6 1
```

![Diagram showing pivot selection process]
EXECUTION EXAMPLE

• Partition, recursive call, pivot selection
EXECUTION EXAMPLE

- Partition, recursive call, base case
EXECUTION EXAMPLE

- Recursive call, …, base case, join
EXECUTION EXAMPLE

• Recursive call, pivot selection
EXECUTION EXAMPLE

- Partition, ..., recursive call, base case

```
2 4 3 1 → 1 2 3 4
7 9 7
```
EXECUTION EXAMPLE

- Join, join

```
7 2 9 4 3 7 6 1 → 1 2 3 4 6 7 7 9
```

```
2 4 3 1 → 1 2 3 4
```

```
7 9 7 → 7 7 9
```

```
1 → 1
```

```
4 3 → 3 4
```

```
9 → 9
```

```
4 → 4
```
WORST-CASE RUNNING TIME

• The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
  • One of $L$ and $G$ has size $n - 1$ and the other has size 0
• The running time is proportional to:
  $n + (n - 1) + \cdots + 2 + 1 = O(n^2)$
• Alternatively, using recurrence equations:
  $T(n) = T(n - 1) + O(n) = O(n^2)$
EXPECTED RUNNING TIME
REMOVING EQUAL SPLIT ASSUMPTION

• Consider a recursive call of quick-sort on a sequence of size $s$
  • Good call: the sizes of $L$ and $G$ are each less than $\frac{3s}{4}$
  • Bad call: one of $L$ and $G$ has size greater than $\frac{3s}{4}$

• A call is good with probability $1/2$
  • $1/2$ of the possible pivots cause good calls:
EXPECTED RUNNING TIME

• **Probabilistic Fact:** The expected number of coin tosses required in order to get \( k \) heads is \( 2k \) (e.g., it is expected to take 2 tosses to get heads)

• For a node of depth \( i \), we expect
  • \( \frac{i}{2} \) ancestors are good calls
  • The size of the input sequence for the current call is at most \( \left( \frac{3}{4} \right)^{\frac{i}{2}} n \)

• Therefore, we have
  • For a node of depth \( 2 \log_{\frac{3}{4}} n \), the expected input size is one
  • The expected height of the quick-sort tree is \( O(\log n) \)

• The amount of work done at the nodes of the same depth is \( O(n) \)

• Thus, the expected running time of quick-sort is \( O(n \log n) \)
**IN-PLACE QUICK-SORT**

- Quick-sort can be implemented to run in-place.
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that:
  - The elements less than the pivot have indices less than $h$.
  - The elements equal to the pivot have indices between $h$ and $k$.
  - The elements greater than the pivot have indices greater than $k$.
- The recursive calls consider:
  - Elements with indices less than $h$.
  - Elements with indices greater than $k$.

**Algorithm** `inPlaceQuickSort(S,l,r)`

**Input:** Array $S$, indices $l,r$

**Output:** Array $S$ with the elements between $l$ and $r$ sorted.

1. if $l \geq r$ then
2.   return $S$
3.   $i \leftarrow \text{rand}() \% (r - l) + l$ //random integer
4.   //between $l$ and $r$
5.   $x \leftarrow S[i]$
6.   $(h,k) \leftarrow \text{inPlacePartition}(x)$
7.   `inPlaceQuickSort(S,l,h - 1)`
8.   `inPlaceQuickSort(S,k + 1,r)`
9.   return $S$
IN-PLACE PARTITIONING

• Perform the partition using two indices to split $S$ into $L$ and $E \cup G$ (a similar method can split $E \cup G$ into $E$ and $G$).

$\begin{array}{c}
3 & 2 & 5 & 1 & 0 & 7 & 3 & 5 & 9 & 2 & 7 & 9 & 8 & 9 & 7 & 6 & 9 \\
\end{array}$

(pivot = 6)

• Repeat until $j$ and $k$ cross:
  • Scan $j$ to the right until finding an element $\geq x$.
  • Scan $k$ to the left until finding an element $< x$.
  • Swap elements at indices $j$ and $k$
# Summary of Sorting Algorithms (So Far)

<table>
<thead>
<tr>
<th>Algorithm</th>
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SORTING LOWER BOUND
Many sorting algorithms are comparison based.
- They sort by making comparisons between pairs of objects
- Examples: bubble-sort, selection-sort, insertion-sort, heap-sort, merge-sort, quick-sort, ...

Let us therefore derive a lower bound on the running time of any algorithm that uses comparisons to sort $n$ elements, $x_1, x_2, ..., x_n$. 
COUNTING COMPARISONS

• Let us just count comparisons then.

• Each possible run of the algorithm corresponds to a root-to-leaf path in a decision tree.
The height of the decision tree is a lower bound on the running time.

Every input permutation must lead to a separate leaf output.

If not, some input \( \ldots 4 \ldots 5 \ldots \) would have the same output ordering as \( \ldots 5 \ldots 4 \ldots \), which would be wrong.

Since there are \( n! = 1 \times 2 \times \ldots \times n \) leaves, the height is at least \( \log(n!) \).
THE LOWER BOUND

• Any comparison-based sorting algorithm takes at least $\log(n!)$ time

$$\log(n!) \geq \log \left(\frac{n}{2}\right)^\frac{n}{2} = \frac{n}{2} \log \frac{n}{2}$$

• That is, any comparison-based sorting algorithm must run in $\Omega(n \log n)$ time.
BUCKET-SORT AND RADIX-SORT

CAN WE SORT IN LINEAR TIME?

B

1, c
3, a
3, b
7, d
7, g
7, e

0 1 2 3 4 5 6 7 8 9
Let $S$ be a sequence of $n$ (key, element) items with keys in the range $[0, N - 1]$

Bucket-sort uses the keys as indices into an auxiliary array $B$ of sequences (buckets)

- Phase 1: Empty sequence $S$ by moving each entry into its bucket $B[k]$
- Phase 2: for $i ← 0 \ldots N - 1$, move the items of bucket $B[i]$ to the end of sequence $S$

Analysis:
- Phase 1 takes $O(n)$ time
- Phase 2 takes $O(n + N)$ time

Bucket-sort takes $O(n + N)$ time

Algorithm $\text{bucketSort}(S, N)$

Input: Sequence $S$ of entries with integer keys in the range $[0, N - 1]$

Output: Sequence $S$ sorted in nondecreasing order of the keys

1. $B ←$ array of $N$ empty sequences
2. for each entry $e ∈ S$ do
3. \qquad $k ← e.key()$
4. \qquad remove $e$ from $S$
5. \qquad insert $e$ at the end of bucket $B[k]$
6. for $i ← 0 \ldots N - 1$ do
7. \qquad for each entry $e ∈ B[i]$ do
8. \qquad remove $e$ from bucket $B[i]$
9. \qquad insert $e$ at the end of $S$
Example

• Key range [37, 46] – map to buckets [0,9]

Phase 1

Phase 2
PROPERTIES AND EXTENSIONS

• Properties
  • Key-type
    • The keys are used as indices into an array and cannot be arbitrary objects
  • No external comparator
  • Stable sorting
    • The relative order of any two items with the same key is preserved after the execution of the algorithm

• Extensions
  • Integer keys in the range $[a, b]$
    • Put entry $e$ into bucket $B[k - a]$
  • String keys from a set $D$ of possible strings, where $D$ has constant size (e.g., names of the 50 U.S. states)
    • Sort $D$ and compute the index $i(k)$ of each string $k$ of $D$ in the sorted sequence
    • Put item $e$ into bucket $B[i(k)]$
LEXICOGRAPHIC ORDER

• Given a list of tuples:
  (7,4,6) (5,1,5) (2,4,6) (2,1,4) (5,1,6) (3,2,4)

• After sorting, the list is in lexicographical order:
  (2,1,4) (2,4,6) (3,2,4) (5,1,5) (5,1,6) (7,4,6)
A $d$-tuple is a sequence of $d$ keys $(k_1, k_2, \ldots, k_d)$, where key $k_i$ is said to be the $i$-th dimension of the tuple.

- Example - the Cartesian coordinates of a point in space is a 3-tuple $(x, y, z)$

The lexicographic order of two $d$-tuples is recursively defined as follows:

$$(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d) \iff$$

$$x_1 < y_1 \lor (x_1 = y_1 \land (x_2, \ldots, x_d) < (y_2, \ldots, y_d))$$

i.e., the tuples are compared by the first dimension, then by the second dimension, etc.
EXERCISE
LEXICOGRAPHIC ORDER

• Given a list of 2-tuples, we can order the tuples lexicographically by applying a stable sorting algorithm two times:
  (3,3) (1,5) (2,5) (1,2) (2,3) (1,7) (3,2) (2,2)

• Possible ways of doing it:
  • Sort first by 1st element of tuple and then by 2nd element of tuple
  • Sort first by 2nd element of tuple and then by 1st element of tuple

• Show the result of sorting the list using both options
EXERCISE
LEXICOGRAPHIC ORDER

• (3,3) (1,5) (2,5) (1,2) (2,3) (1,7) (3,2) (2,2)
• Using a stable sort,
  • Sort first by 1st element of tuple and then by 2nd element of tuple
  • Sort first by 2nd element of tuple and then by 1st element of tuple
• Option 1:
  • 1st sort: (1,5) (1,2) (1,7) (2,5) (2,3) (2,2) (3,3) (3,2)
  • 2nd sort: (1,2) (2,2) (3,2) (2,3) (3,3) (1,5) (2,5) (1,7) - WRONG
• Option 2:
  • 1st sort: (1,2) (3,2) (2,2) (3,3) (2,3) (1,5) (2,5) (1,7)
  • 2nd sort: (1,2) (1,5) (1,7) (2,2) (2,3) (2,5) (3,2) (3,3) - CORRECT
**LEXICOGRAPHIC-SORT**

- Let $C_i$ be the comparator that compares two tuples by their $i$-th dimension
- Let $\text{stableSort}(S, C)$ be a stable sorting algorithm that uses comparator $C$
- Lexicographic-sort sorts a sequence of $d$-tuples in lexicographic order by executing $d$ times algorithm $\text{stableSort}$, one per dimension
- Lexicographic-sort runs in $O(dT(n))$ time, where $T(n)$ is the running time of $\text{stableSort}$

**Algorithm** $\text{lexicographicSort}(S)$

**Input:** Sequence $S$ of $d$-tuples

**Output:** Sequence $S$ sorted in lexicographic order

1. for $i \leftarrow d \ldots 1$ do
2. $\text{stableSort}(S, C_i)$
**RADIX-SORT**

- Radix-sort is a specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension.
- Radix-sort is applicable to tuples where the keys in each dimension \( i \) are integers in the range \([0, N - 1]\).
- Radix-sort runs in time \( O(d(n + N)) \)

**Algorithm** \( \text{radixSort}(S, N) \)

**Input:** Sequence \( S \) of \( d \)-tuples such that 
\[
(0, ..., 0) \leq (x_1, ..., x_d) \quad \text{and} \\
(x_1, ..., x_d) \leq (N - 1, ..., N - 1)
\]
for each tuple \((x_1, ..., x_d)\) in \( S \)

**Output:** Sequence \( S \) sorted in lexicographic order

1. for \( i \leftarrow d \ldots 1 \) do
2. set the key \( k \) of each entry \((k, (x_1, ..., x_d))\) of \( S \) to \( i \)th dimension \( x_i \)
3. \( \text{bucketSort}(S, N) \)
EXAMPLE
RADIX-SORT FOR BINARY NUMBERS
• Sorting a sequence of 4-bit integers
  • $d = 4, N = 2$ so $O(d(n + N)) = O(4(n + 2)) = O(n)$

```plaintext
1001
0010
1101
0001
1110
0010
1101
0001
1101
1001
1101
0001
1110
1001
0010
1101
1110
0001
1001
0010
1101
1110
1001
0010
1101
1110
```
## SUMMARY OF SORTING ALGORITHMS

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<td>Radix Sort</td>
<td>$O(d(n + N))$, $d$ #digits, $N$ range of digit values</td>
<td>Stable</td>
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<td>Fastest, only for integers</td>
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THE SELECTION PROBLEM

• Given an integer \( k \) and \( n \) elements \( \{x_1, x_2, \ldots, x_n\} \), taken from a total order, find the \( k \)-th smallest element in this set.
  • Also called order statistics, the \( i \)th order statistic is the \( i \)th smallest element
  • Minimum - \( k = 1 \) - 1st order statistic
  • Maximum - \( k = n \) - \( n \)th order statistic
  • Median - \( k = \left\lceil \frac{n}{2} \right\rceil \)
  • etc
THE SELECTION PROBLEM

• Naïve solution - SORT!

• We can sort the set in $O(n \log n)$ time and then index the $k$-th element.

• Can we solve the selection problem faster?

7 4 9 6 2 → 2 4 6 7 9

k=3
THE MINIMUM (OR MAXIMUM)

Algorithm minimum(A)
Input: Array A
Output: minimum element in A
1. $m \leftarrow A[1]
2. for $i \leftarrow 2 \ldots n$ do
3. \hspace{1em} $m \leftarrow \min(m, A[i])$
4. return $m$

• Running Time
  • $O(n)$

• Is this the best possible?
QUICK-SELECT

• **Quick-select** is a randomized selection algorithm based on the prune-and-search paradigm:
  • **Prune**: pick a random element \( x \) (called pivot) and partition \( S \) into
    • \( L \) elements \(< x\)
    • \( E \) elements \(= x\)
    • \( G \) elements \( > x \)
  • **Search**: depending on \( k \), either answer is in \( E \), or we need to recur on either \( L \) or \( G \)
• **Note**: Partition same as Quicksort

\[
\begin{align*}
|L| < k & \leq |L| + |E| \quad \text{(done)} \\
k' & = k - |L| - |E| \\
k > |L| + |E| & \quad k' = k - |L| - |E|
\end{align*}
\]
QUICK-SELECT VISUALIZATION

- An execution of quick-select can be visualized by a recursion path
  - Each node represents a recursive call of quick-select, and stores $k$ and the remaining sequence

- $k = 5, S = (7, 4, 9, 3, 2, 6, 5, 1, 8)$
- $k = 2, S = (7, 4, 9, 6, 5, 8)$
- $k = 2, S = (7, 4, 6, 5)$
- $k = 1, S = (7, 6, 5)$
- 5
EXERCISE

• Best Case - even splits (n/2 and n/2)
• Worst Case - bad splits (1 and n-1)

• Derive and solve the recurrence relation corresponding to the best case performance of randomized quick-select.
• Derive and solve the recurrence relation corresponding to the worst case performance of randomized quick-select.

Good call

Bad call
EXPECTED RUNNING TIME

• Consider a recursive call of quick-select on a sequence of size $S$
  • Good call: the size of $L$ and $G$ is at most $\frac{3S}{4}$
  • Bad call: the size of $L$ and $G$ is greater than $\frac{3S}{4}$

• A call is good with probability $1/2$
  • $1/2$ of the possible pivots cause good calls:

1. Good call
   - $2 4 3 1$
   - $7 9 7 1 \rightarrow 1$

2. Bad call
   - $7 2 9 4 3 7 6 1$
   - $1$
   - $7 2 9 4 3 7 6$

Good pivots: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16
Bad pivots: 12, 13, 14, 15, 16
EXPECTED RUNNING TIME

• Probabilistic Fact #1: The expected number of coin tosses required in order to get one head is two.

• Probabilistic Fact #2: Expectation is a linear function:
  • \( E(X + Y) = E(X) + E(Y) \)
  • \( E(cX) = cE(X) \)

• Let \( T(n) \) denote the expected running time of quick-select.

• By Fact #2, \( T(n) < T\left(\frac{3n}{4}\right) + bn \) * (expected # of calls before a good call)

• By Fact #1, \( T(n) < T\left(\frac{3n}{4}\right) + 2bn \)

• That is, \( T(n) \) is a geometric series: \( T(n) < 2bn + 2b \left(\frac{3}{4}\right) n + 2b \left(\frac{3}{4}\right)^2 n + 2b \left(\frac{3}{4}\right)^3 n + \ldots \)

• So \( T(n) \) is \( O(n) \).

• We can solve the selection problem in \( O(n) \) expected time.
DETERMINISTIC SELECTION

• We can do selection in \( O(n) \) worst-case time.

• Main idea: recursively use the selection algorithm itself to find a good pivot for quick-select:
  • Divide \( S \) into \( \frac{n}{5} \) sets of 5 each
  • Find a median in each set
  • Recursively find the median of the “baby” medians.

• See Exercise C-12.56 for details of analysis.
INTERVIEW QUESTION 1

• You are given two sorted arrays, $A$ and $B$, where $A$ has a large enough buffer at the end to hold $B$. Write a method to merge $B$ into $A$ in sorted order.
INTERVIEW QUESTION 2

• Write a method to sort an array of strings so that all the anagrams are next to each other.
  • Two words are anagrams if they use the exact same letters, i.e., race and care are anagrams

INTERVIEW QUESTION 3

• Imagine you have a 2 TB file with one string per line. Explain how you would sort the file.