CHAPTER 10
MAPS, HASH TABLES, AND SKIP LISTS

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MAPS

- A map models a searchable collection of key-value entries
- The main operations of a map are for searching, inserting, and deleting items
- Multiple entries with the same key are **not** allowed
- Applications:
  - address book
  - student-record database
- Often called **associative containers**
THE MAP ADT

- **get(k):** if the map \( M \) has an entry with key \( k \), return its associated value; else, return null
- **put(k, v):** insert entry \((k, v)\) into the map \( M \); if key \( k \) is not already in \( M \), then return null; else, return old value associated with \( k \)
- **remove(k):** if the map \( M \) has an entry with key \( k \), remove it from \( M \) and return its associated value; else, return null
- **size(), isEmpty()**
- **entrySet():** return an iterable collection of the entries in \( M \)
- **keySet():** return an iterable collection of the keys in \( M \)
- **values():** return an iterator of the values in \( M \)
Example

<table>
<thead>
<tr>
<th>Operation</th>
<th>Output</th>
<th>Map</th>
</tr>
</thead>
<tbody>
<tr>
<td>isEmpty()</td>
<td>true</td>
<td>Ø</td>
</tr>
<tr>
<td>put(5,A)</td>
<td>null</td>
<td>(5,A)</td>
</tr>
<tr>
<td>put(7,B)</td>
<td>null</td>
<td>(5,A), (7,B)</td>
</tr>
<tr>
<td>put(2,C)</td>
<td>null</td>
<td>(5,A), (7,B), (2,C)</td>
</tr>
<tr>
<td>put(8,D)</td>
<td>null</td>
<td>(5,A), (7,B), (2,C), (8,D)</td>
</tr>
<tr>
<td>put(2,E)</td>
<td>C</td>
<td>(5,A), (7,B), (2,E), (8,D)</td>
</tr>
<tr>
<td>get(7)</td>
<td>B</td>
<td>(5,A), (7,B), (2,E), (8,D)</td>
</tr>
<tr>
<td>get(4)</td>
<td>null</td>
<td>(5,A), (7,B), (2,E), (8,D)</td>
</tr>
<tr>
<td>get(2)</td>
<td>E</td>
<td>(5,A), (7,B), (2,E), (8,D)</td>
</tr>
<tr>
<td>size()</td>
<td>4</td>
<td>(5,A), (7,B), (2,E), (8,D)</td>
</tr>
<tr>
<td>remove(5)</td>
<td>A</td>
<td>(7,B), (2,E), (8,D)</td>
</tr>
<tr>
<td>remove(2)</td>
<td>E</td>
<td>(7,B), (8,D)</td>
</tr>
<tr>
<td>get(2)</td>
<td>null</td>
<td>(7,B), (8,D)</td>
</tr>
<tr>
<td>isEmpty()</td>
<td>false</td>
<td>(7,B), (8,D)</td>
</tr>
</tbody>
</table>
LIST-BASED MAP

- We can implement a map with an unsorted list
  - Store the entries in arbitrary order
- Complexity of get, put, remove?
  - $O(n)$ on put, get, and remove
DIRECT ADDRESS TABLE MAP IMPLEMENTATION

• A direct address table is a map in which
  • The keys are in the range \([0, N]\)
  • Stored in an array \(T\) of size \(N\)
  • Entry with key \(k\) stored in \(T[k]\)

• Performance:
  • \(\text{put}(k, v), \text{get}(k), \text{and remove}(k)\) all take \(O(1)\) time
  • Space - requires space \(O(N)\), independent of \(n\), the number of entries stored in the map

• The direct address table is not space efficient unless the range of the keys is close to the number of elements to be stored in the map, i.e., unless \(n\) is close to \(N\).
SORTED MAP

• A Sorted Map supports the usual map operations, but also maintains an order relation for the keys.

• Naturally supports
  • Sorted search tables - store dictionary in an array by non-decreasing order of the keys
  • Utilizes binary search

• Sorted Map ADT adds the following functionality to a map
  • `firstEntry()`, `lastEntry()` – return iterators to entries with the smallest and largest keys, respectively
  • `ceilingEntry(k)`, `floorEntry(k)` – return an iterator to the least/greatest key value greater than/less than or equal to \( k \)
  • `lowerEntry(k)`, `higherEntry(k)` – return an iterator to the greatest/least key value less than/greater than \( k \)
  • etc
EXAMPLE OF ORDERED MAP: BINARY SEARCH

• Binary search performs operation $\text{get}(k)$ on an ordered search table
  • similar to the high-low game
  • at each step, the number of candidate items is halved
  • terminates after a logarithmic number of steps

• Example

$\text{get}(7)$
# Simple Map Implementation Summary

<table>
<thead>
<tr>
<th></th>
<th>( \text{put}(k, v) )</th>
<th>( \text{get}(k) )</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unsorted list</td>
<td>( O(n) )</td>
<td>( O(n) )</td>
<td>( O(n) )</td>
</tr>
<tr>
<td>Direct Address Table</td>
<td>( O(1) )</td>
<td>( O(1) )</td>
<td>( O(N) )</td>
</tr>
<tr>
<td>Sorted Search Table</td>
<td>( O(n) )</td>
<td>( O(\log n) )</td>
<td>( O(n) )</td>
</tr>
<tr>
<td>(Naturally supported Sorted Map)</td>
<td>( O(n) )</td>
<td>( O(\log n) )</td>
<td>( O(n) )</td>
</tr>
</tbody>
</table>
DEFINITIONS

• A **set** is an unordered collection of elements, without duplicates that typically supports efficient membership tests.
  • Elements of a set are like keys of a map, but without any auxiliary values.

• A **multiset** (also known as a bag) is a set-like container that allows duplicates.

• A **multimap** (also known as a dictionary) is similar to a traditional map, in that it associates values with keys; however, in a multimap the same key can be mapped to multiple values.
  • For example, the index of a book maps a given term to one or more locations at which the term occurs.
**SET ADT**

**add(e):** Adds the element $e$ to $S$ (if not already present).

**remove(e):** Removes the element $e$ from $S$ (if it is present).

**contains(e):** Returns whether $e$ is an element of $S$.

**iterator():** Returns an iterator of the elements of $S$.

There is also support for the traditional mathematical set operations of **union**, **intersection**, and **subtraction** of two sets $S$ and $T$:

- $S \cup T = \{ e : e \text{ is in } S \text{ or } e \text{ is in } T \}$,
- $S \cap T = \{ e : e \text{ is in } S \text{ and } e \text{ is in } T \}$,
- $S - T = \{ e : e \text{ is in } S \text{ and } e \text{ is not in } T \}$.

**addAll(T):** Updates $S$ to also include all elements of set $T$, effectively replacing $S$ by $S \cup T$.

**retainAll(T):** Updates $S$ so that it only keeps those elements that are also elements of set $T$, effectively replacing $S$ by $S \cap T$.

**removeAll(T):** Updates $S$ by removing any of its elements that also occur in set $T$, effectively replacing $S$ by $S - T$. 
**GENERIC MERGING**

- Generalized merge of two sorted lists A and B
- Template method `genericMerge`
- Auxiliary methods
  - `aIsLess`
  - `bIsLess`
  - `bothAreEqual`
- Runs in $O(n_A + n_B)$ time provided the auxiliary methods run in $O(1)$ time

```
Algorithm genericMerge(A, B)
Input: Sets A, B as sorted lists
Output: Set S
1. $S \leftarrow \emptyset$
2. while $\neg A.isEmpty() \land \neg B.isEmpty()$ do
3.   $a \leftarrow A.first(); b \leftarrow B.first()$
4.   if $a < b$
5.     aIsLess(a, S) //generic action
6.     A.removeFirst();
7.   else if $b < a$
8.     bIsLess(b, S) //generic action
9.     B.removeFirst();
10.  else // $a = b$
11.   bothAreEqual(a, b, S) //generic action
12.   A.removeFirst(); B.removeFirst()
13. while $\neg A.isEmpty()$ do
14.   aIsLess(A.first(), S); A.eraseFront()
15. while $\neg B.isEmpty()$ do
16.   bIsLess(B.first(), S); B.removeFirst()
17. return S
```
USING GENERIC MERGE FOR SET OPERATIONS

• Any of the set operations can be implemented using a generic merge.

• For example:
  • For intersection: only copy elements that are duplicated in both list.
  • For union: copy every element from both lists except for the duplicates.

• All methods run in linear time.
MULTIMAP

• A **multimap** is similar to a map, except that it can store multiple entries with the same key

• We can implement a multimap $M$ by means of a map $M'$
  • For every key $k$ in $M$, let $E(k)$ be the list of entries of $M$ with key $k$
  • The entries of $M'$ are the pairs $(k, E(k))$
**MULTIMAPS**

- **get(k)**: Returns a collection of all values associated with key k in the multimap.
- **put(k, v)**: Adds a new entry to the multimap associating key k with value v, without overwriting any existing mappings for key k.
- **remove(k, v)**: Removes an entry mapping key k to value v from the multimap (if one exists).
- **removeAll(k)**: Removes all entries having key equal to k from the multimap.
- **size()**: Returns the number of entries of the multiset (including multiple associations).
- **entries()**: Returns a collection of all entries in the multimap.
- **keys()**: Returns a collection of keys for all entries in the multimap (including duplicates for keys with multiple bindings).
- **keySet()**: Returns a nonduplicative collection of keys in the multimap.
- **values()**: Returns a collection of values for all entries in the multimap.
HASH TABLES
INTUITIVE NOTION OF A MAP

• Intuitively, a map $M$ supports the abstraction of using keys as indices with a syntax such as $M[k]$.

• As a mental warm-up, consider a restricted setting in which a map with $n$ items uses keys that are known to be integers in a range from 0 to $N - 1$, for some $N \geq n$. 

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
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<th>6</th>
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<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>D</td>
<td></td>
<td>Z</td>
<td></td>
<td></td>
<td>C</td>
<td>Q</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
MORE GENERAL KINDS OF KEYS

• But what should we do if our keys are not integers in the range from 0 to $N - 1$?
  • Use a hash function to map general keys to corresponding indices in a table.
  • For instance, the last four digits of a Social Security number.
HASH TABLES

• A Hash function $h(k) \rightarrow [0, N - 1]$
  • The integer $h(k)$ is referred to as the hash value of key $k$
  • Example - $h(k) = k \mod N$ could be a hash function for integers

• Hash tables consist of
  • A hash function $h$
  • Array $A$ of size $N$ (either to an element itself or to a “bucket”)

• Goal is to store elements $(k, v)$ at index $i = h(k)$
ISSUES WITH HASH TABLES

• Issues

  • Collisions - some keys will map to the same index of H (otherwise we have a Direct Address Table).
    • Chaining - put values that hash to same location in a linked list (or a “bucket”)
    • Open addressing - if a collision occurs, have a method to select another location in the table.

• Load factor

• Rehashing
EXAMPLE

• We design a hash table for a Map storing items (SSN, Name), where SSN (social security number) is a nine-digit positive integer

• Our hash table uses an array of size \( N = 10,000 \) and the hash function \( h(k) = \text{last four digits of } k \)
A hash function is usually specified as the composition of two functions:

**Hash code:**
\[ h_1 : \text{keys} \rightarrow \text{integers} \]

**Compression function:**
\[ h_2 : \text{integers} \rightarrow [0, N - 1] \]

The hash code is applied first, and the compression function is applied next on the result, i.e.,
\[ h(k) = h_2(h_1(k)) \]

The goal of the hash function is to “disperse” the keys in an apparently random way.
HASH CODES

• **Memory address:**
  - We reinterpret the memory address of the key object as an integer
  - Good in general, except for numeric and string keys

• **Integer cast:**
  - We reinterpret the bits of the key as an integer
  - Suitable for keys of length less than or equal to the number of bits of the integer type (e.g., byte, short, int and float in C++)

• **Component sum:**
  - We partition the bits of the key into components of fixed length (e.g., 16 or 32 bits) and we sum the components (ignoring overflows)
  - Suitable for numeric keys of fixed length greater than or equal to the number of bits of the integer type (e.g., long and double in C++)
HASH CODES

• **Polynomial accumulation:**
  • We partition the bits of the key into a sequence of components of fixed length (e.g., 8, 16 or 32 bits)
    \[ a_0 a_1 \ldots a_{n-1} \]
  • We evaluate the polynomial
    \[ p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} \]
    at a fixed value \( z \), ignoring overflows
  • Especially suitable for strings (e.g., the choice \( z = 33 \) gives at most 6 collisions on a set of 50,000 English words)

• **Cyclic Shift:**
  • Like polynomial accumulation except use bit shifts instead of multiplications and bitwise or instead of addition
  • Can be used on floating point numbers as well by converting the number to an array of characters
COMPRESSION FUNCTIONS

• Division:
  • $h_2(k) = k \mod N$
  • The size $N$ of the hash table is usually chosen to be a prime
  • The reason has to do with number theory and is beyond the scope of this course

• Multiply, Add and Divide (MAD):
  • $h_2(k) = (ak + b) \mod N$
  • $a$ and $b$ are nonnegative integers such that
    $$a \mod N \neq 0$$
  • Otherwise, every integer would map to the same value $b$
COLLISION RESOLUTION WITH SEPARATE CHAINING

- **Collisions** occur when different elements are mapped to the same cell.
- **Separate Chaining**: let each cell in the table point to a linked list of entries that map there.

- Chaining is simple, but requires additional memory outside the table.

```
0 1 2 3 4
∅ 025-612-0001 ∅ ∅
∅ ∅ 451-229-0004 ∅
∅ ∅ ∅ 981-101-0004
```
EXERCISE
SEPARATE CHAINING

• Assume you have a hash table $H$ with $N = 9$ slots ($A[0 - 8]$) and let the hash function be $h(k) = k \mod N$

• Demonstrate (by picture) the insertion of the following keys into a hash table with collisions resolved by chaining
  • 5, 28, 19, 15, 20, 33, 12, 17, 10
In **Open addressing** the colliding item is placed in a different cell of the table.

**Linear probing** handles collisions by placing the colliding item in the next (circularly) available table cell. So the $i$th cell checked is:

$$h(k, i) = [h(k) + i] \mod N$$

Each table cell inspected is referred to as a "probe".

Colliding items lump together, causing future collisions to cause a longer probe sequence.

**Example:**

- $h(k) = k \mod 13$
- Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order.

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
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<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>41</td>
<td></td>
<td></td>
<td></td>
<td>18</td>
<td>44</td>
<td>59</td>
<td>32</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td></td>
<td>31</td>
<td>73</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>
Consider a hash table $A$ that uses linear probing

**get** $(k)$

- We start at cell $h(k)$
- We probe consecutive locations until one of the following occurs
  - An item with key $k$ is found, or
  - An empty cell is found, or
  - $N$ cells have been unsuccessfully probed

**Algorithm get** $(k)$

1. $i \leftarrow h(k)$
2. $p \leftarrow 0$
3. repeat
   4. $c \leftarrow A[i]$
   5. if $c \neq \emptyset$
   6. return null
   7. else if $c.key() = k$
   8. return $c$
   9. else
   10. $i \leftarrow (i + 1) \mod N$
   11. $p \leftarrow p + 1$
12. until $p = N$
13. return null
**UPDATES WITH LINEAR PROBING**

- To handle insertions and deletions, we introduce a special object, called **DEFUNCT**, which replaces deleted elements.

- **remove(k)**
  - We search for an item with key $k$.
  - If such an item $(k, v)$ is found, we replace it with the special item DEFUNCT.
  - Else, we return null.

- **put(k, v)**
  - We start at cell $h(k)$.
  - We probe consecutive cells until one of the following occurs:
    - A cell $i$ is found that is either empty or stores DEFUNCT, or
    - $N$ cells have been unsuccessfully probed.
EXERCISE
OPEN ADDRESSING – LINEAR PROBING

• Assume you have a hash table $H$ with $N = 11$ slots ($A[0 \rightarrow 10]$) and let the hash function be $h(k) = k \mod N$

• Demonstrate (by picture) the insertion of the following keys into a hash table with collisions resolved by linear probing.
  • 10, 22, 31, 4, 15, 28, 17, 88, 59
COLLISION RESOLUTION WITH OPEN ADDRESSING – QUADRATIC PROBING

- Linear probing has an issue with clustering.
- Another strategy called quadratic probing uses a hash function
  \[ h(k, i) = (h(k) + i^2) \mod N \]
  for \( i = 0, 1, ..., N - 1 \)
- This can still cause secondary clustering.
Double hashing uses a secondary hash function \( h_2(k) \) and handles collisions by placing an item in the first available cell of the series

\[
h(k, i) = (h_1(k) + ih_2(k)) \mod N
\]

for \( i = 0, 1, \ldots, N - 1 \)

- The secondary hash function \( h_2(k) \) cannot have zero values
- The table size \( N \) must be a prime to allow probing of all the cells

Common choice of compression map for the secondary hash function:

\[
h_2(k) = q - (k \mod q)
\]

where

- \( q < N \)
- \( q \) is a prime

- The possible values for \( h_2(k) \) are \( 1, 2, \ldots, q \)
PERFORMANCE OF HASHING

- In the worst case, searches, insertions and removals on a hash table take $O(n)$ time.
- The worst case occurs when all the keys inserted into the map collide.
- The load factor $\alpha = \frac{n}{N}$ affects the performance of a hash table.
- Assuming that the hash values are like random numbers, it can be shown that the expected number of probes for an insertion with open addressing is:
  $$\frac{1}{1 - \alpha} = \frac{1}{1 - n/N} = \frac{1}{N - n/N} = \frac{1}{N - n}$$
- The expected running time of all the Map ADT operations in a hash table is $O(1)$.
- In practice, hashing is very fast provided the load factor is not close to 100%.
- Applications of hash tables:
  - Small databases
  - Compilers
  - Browser caches
UNIFORM HASHING ASSUMPTION

• The probe sequence of a key $k$ is the sequence of slots probed when looking for $k$
  • In open addressing, the probe sequence is $h(k, 0), h(k, 1), \ldots, h(k, N - 1)$

• Uniform Hashing Assumption
  • Each key is equally likely to have any one of the $N!$ permutations of $\{0, 1, \ldots, N - 1\}$ as is the probe sequence
  • Note: Linear probing and double hashing are far from achieving Uniform Hashing
    • Linear probing: $N$ distinct probe sequences
    • Double Hashing: $N^2$ distinct probe sequences
PERFORMANCE OF UNIFORM HASHING

• Theorem: Assuming uniform hashing and an open-address hash table with load factor $\alpha = \frac{n}{N} < 1$, the expected number of probes in an unsuccessful search is at most $\frac{1}{1-\alpha}$.

• Exercise: compute the expected number of probes in an unsuccessful search in an open address hash table with $\alpha = \frac{1}{2}$, $\alpha = \frac{3}{4}$, and $\alpha = \frac{99}{100}$. 
ON REHASHING

• Keeping the load factor low is vital for performance

• When resizing the table:
  • Reallocate space for the array (of size that is a prime)
  • Design a new hash function (new parameters) for the new array size (practically, change the mod)
  • For each item you reinsert into the table rehash
<table>
<thead>
<tr>
<th>Method</th>
<th>put($k, v$)</th>
<th>get($k$)</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unsorted list</strong></td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td><strong>Direct Address Table</strong></td>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(N)$</td>
</tr>
<tr>
<td><strong>Sorted Search Table</strong></td>
<td>$O(n)$</td>
<td>$O(\log n)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>(Naturally supported Sorted Map)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Hashing</strong> (chaining)</td>
<td>$O\left(\frac{n}{N}\right)$</td>
<td>$O\left(\frac{n}{N}\right)$</td>
<td>$O(n + N)$</td>
</tr>
<tr>
<td><strong>Hashing</strong> (open addressing)</td>
<td>$O\left(\frac{1}{1 - \frac{n}{N}}\right)$</td>
<td>$O\left(\frac{1}{1 - \frac{n}{N}}\right)$</td>
<td>$O(N)$</td>
</tr>
</tbody>
</table>
SKIP LISTS
A randomized algorithm controls its execution through random selection (e.g., coin tosses).

It contains statements like:

```plaintext
b ← randomBit()
if b = 0
  do something...
else // b = 1
  do something else...
```

Its running time depends on the outcomes of the “coin tosses”.

Through probabilistic analysis we can derive the expected running time of a randomized algorithm.

We make the following assumptions in the analysis:

- the coins are unbiased
- the coin tosses are independent

The worst-case running time of a randomized algorithm is often large but has very low probability (e.g., it occurs when all the coin tosses give “heads”).

We use a randomized algorithm to insert items into a skip list to insert in expected $O(\log n)$—time.

When randomization is used in data structures they are referred to as probabilistic data structures.
WHAT IS A SKIP LIST?

A skip list for a set $S$ of distinct (key, element) items is a series of lists $S_0, S_1, \ldots, S_h$

- Each list $S_i$ contains the special keys $+\infty$ and $-\infty$
- List $S_0$ contains the keys of $S$ in non-decreasing order
- Each list is a subsequence of the previous one, i.e., $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$
- List $S_h$ contains only the two special keys

Skip lists are one way to implement the Ordered Map ADT

Java applet
IMPLEMENTATION

• We can implement a skip list with quad-nodes
• A quad-node stores:
  • (Key, Value)
  • links to the nodes before, after, below, and above
• Also, we define special keys $+\infty$ and $-\infty$, and we modify the key comparator to handle them
SEARCH - GET (K)

• We search for a key \( k \) in a skip list as follows:
  • We start at the first position of the top list
  • At the current position \( p \), we compare \( k \) with \( y \leftarrow p.next().key() \)
    \( x = y \): we return \( p.next().value() \)
    \( x > y \): we scan forward
    \( x < y \): we drop down
  • If we try to drop down past the bottom list, we return null

• Example: search for 78
EXERCISE
SEARCH

• We search for a key $k$ in a skip list as follows:
  • We start at the first position of the top list
  • At the current position $p$, we compare $k$ with $y \leftarrow p$.next().key()
    $x = y$: we return $p$.next().value()
    $x > y$: we scan forward
    $x < y$: we drop down
  • If we try to drop down past the bottom list, we return NO_SUCH_KEY

• Ex 1: search for 64: list the $(S_i, \text{node})$ pairs visited and the return value
• Ex 2: search for 27: list the $(S_i, \text{node})$ pairs visited and the return value

\[ S_3 \quad -\infty \quad \cdots \quad +\infty \]
\[ S_2 \quad -\infty \quad 31 \quad \cdots \quad +\infty \]
\[ S_1 \quad -\infty \quad 23 \quad 31 \quad 34 \quad 64 \quad \cdots \quad +\infty \]
\[ S_0 \quad -\infty \quad 12 \quad 23 \quad 26 \quad 31 \quad 34 \quad 44 \quad 56 \quad 64 \quad 78 \quad +\infty \]
**INSERTION - PUT \((k, v)\)**

- To insert an item \((k, v)\) into a skip list, we use a randomized algorithm:
  - We repeatedly toss a coin until we get tails, and we denote with \(i\) the number of times the coin came up heads
  - If \(i \geq h\), we add to the skip list new lists \(S_{h+1}, ..., S_{i+1}\) each containing only the two special keys
  - We search for \(k\) in the skip list and find the positions \(p_0, p_1, ..., p_i\) of the items with largest key less than \(k\) in each list \(S_0, S_1, ..., S_i\)
  - For \(i \leftarrow 0, ..., i\), we insert item \((k, v)\) into list \(S_i\) after position \(p_i\)

- Example: insert key 15, with \(i = 2\)
DELETION - REMOVE (K)

• To remove an item with key $k$ from a skip list, we proceed as follows:
  • We search for $k$ in the skip list and find the positions $p_0, p_1, \ldots, p_i$ of the items with key $k$, where position $p_i$ is in list $S_i$
  • We remove positions $p_0, p_1, \ldots, p_i$ from the lists $S_0, S_1, \ldots, S_i$
  • We remove all but one list containing only the two special keys

• Example: remove key 34
SPACE USAGE

• The space used by a skip list depends on the random bits used by each invocation of the insertion algorithm
• We use the following two basic probabilistic facts:
  • Fact 1: The probability of getting \( i \) consecutive heads when flipping a coin is \( \frac{1}{2^i} \)
  • Fact 2: If each of \( n \) items is present in a set with probability \( p \), the expected size of the set is \( np \)

Consider a skip list with \( n \) items
• By Fact 1, we insert an item in list \( S_i \) with probability \( \frac{1}{2^i} \)
• By Fact 2, the expected size of list \( S_i \) is \( \frac{n}{2^i} \)
• The expected number of nodes used by the skip list is
  \[
  \sum_{i=0}^{h} \frac{n}{2^i} = n \sum_{i=0}^{h} \frac{1}{2^i} < 2n
  \]
• Thus the expected space is \( O(2n) \)
The running time of $\text{find}(k)$, $\text{put}(k, v)$, and $\text{erase}(k)$ operations are affected by the height $h$ of the skip list.

We show that with high probability, a skip list with $n$ items has height $O(\log n)$.

We use the following additional probabilistic fact:

- Fact 3: If each of $n$ events has probability $p$, the probability that at least one event occurs is at most $np$.

Consider a skip list with $n$ items:

- By Fact 1, we insert an item in list $S_i$ with probability $\frac{1}{2^i}$.
- By Fact 3, the probability that list $S_i$ has at least one item is at most $\frac{n}{2^i}$.

By picking $i = 3 \log n$, we have that the probability that $S_{3 \log n}$ has at least one item is at most $\frac{n}{2^3 \log n} = \frac{n}{n^3} = \frac{1}{n^2}$.

Thus a skip list with $n$ items has height at most $3 \log n$ with probability at least $1 - \frac{1}{n^2}$. 
SEARCH AND UPDATE TIMES

• The search time in a skip list is proportional to
  • the number of drop-down steps
  • the number of scan-forward steps

• The drop-down steps are bounded by the height of the skip list and thus are $O(\log n)$ expected time

• To analyze the scan-forward steps, we use yet another probabilistic fact:
  • Fact 4: The expected number of coin tosses required in order to get tails is 2

• When we scan forward in a list, the destination key does not belong to a higher list
  • A scan-forward step is associated with a former coin toss that gave tails

• By Fact 4, in each list the expected number of scan-forward steps is 2

• Thus, the expected number of scan-forward steps is $O(\log n)$

• We conclude that a search in a skip list takes $O(\log n)$ expected time

• The analysis of insertion and deletion gives similar results
EXERCISE

• You are working for ObscureDictionaries.com a new online start-up which specializes in sci-fi languages. The CEO wants your team to describe a data structure which will efficiently allow for searching, inserting, and deleting new entries. You believe a skip list is a good idea, but need to convince the CEO. Perform the following:
  • Illustrate insertion of “X-wing” into this skip list. Randomly generated (1, 1, 1, 0).
  • Illustrate deletion of an incorrect entry “Enterprise”
  • Argue the complexity of deleting from a skip list
SUMMARY

- A skip list is a data structure for dictionaries that uses a randomized insertion algorithm.

- In a skip list with $n$ items:
  - The expected space used is $O(n)$.
  - The expected search, insertion and deletion time is $O(\log n)$.

- Using a more complex probabilistic analysis, one can show that these performance bounds also hold with high probability.

- Skip lists are fast and simple to implement in practice.