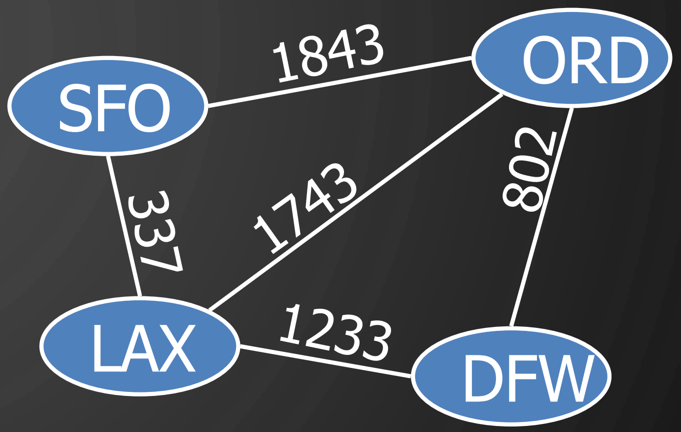
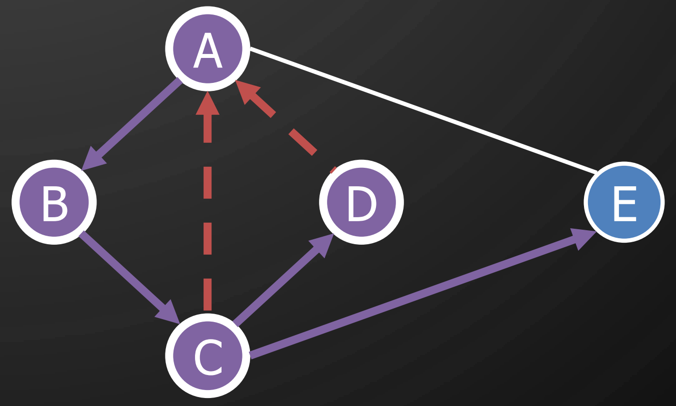
CHAPTER 14
GRAPH ALGORITHMS

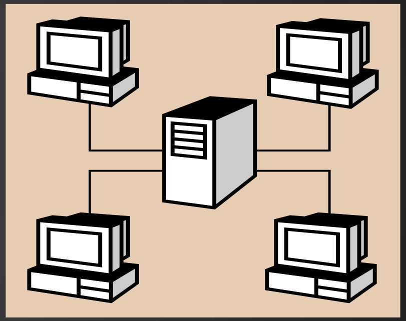


ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN JAVA, GOODRICH, TAMASSIA AND GOLDWASSER (WILEY 2016)

DEPTH-FIRST SEARCH



DEPTH-FIRST SEARCH

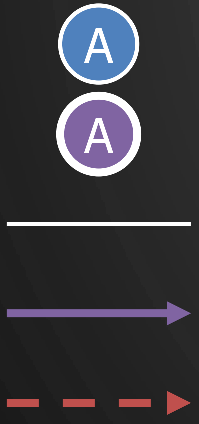


- **Depth-first search (DFS)** is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- DFS on a graph with n vertices and m edges takes $O(n+m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs as what Euler tour is to binary trees

DFS ALGORITHM FROM A VERTEX

```
Algorithm DFS(G, u)
Input: A graph G and a vertex u of G
Output: A collection of vertices reachable from u,
        with their discovery edges
1.  Mark u as visited
2.  for each edge e = (u,v) ∈ G.outgoingEdges(u) do
3.    if v has not been visited then
4.      Record e as a discovery edge for v
5.      DFS(G, v)
```

EXAMPLE



unexplored vertex

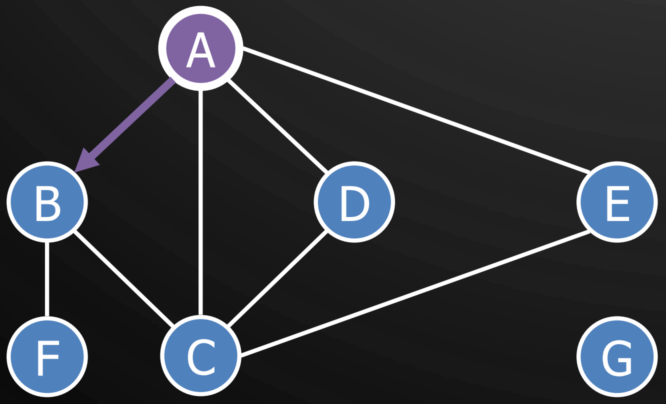
visited vertex

unexplored edge

discovery edge

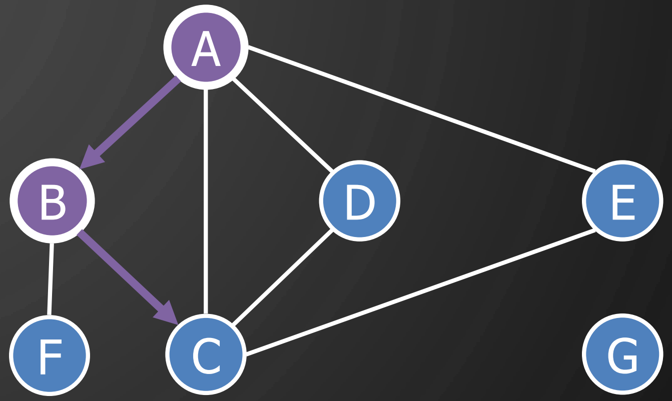
back edge

$I(A) = \{\boldsymbol{B}, C, D, E\}$

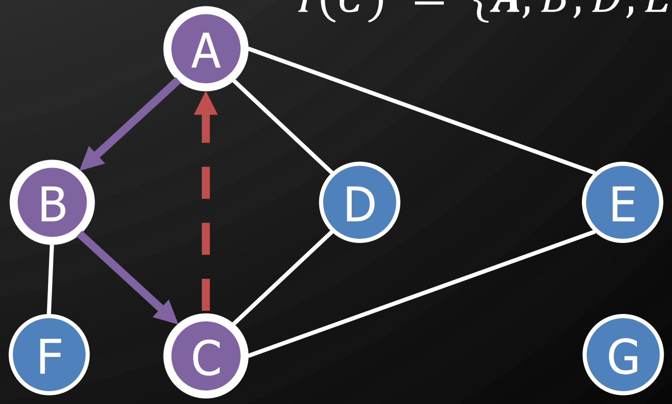


$I(B) = \{\boldsymbol{A}, C, F\}$

$I(B) = \{A, \boldsymbol{C}, F\}$



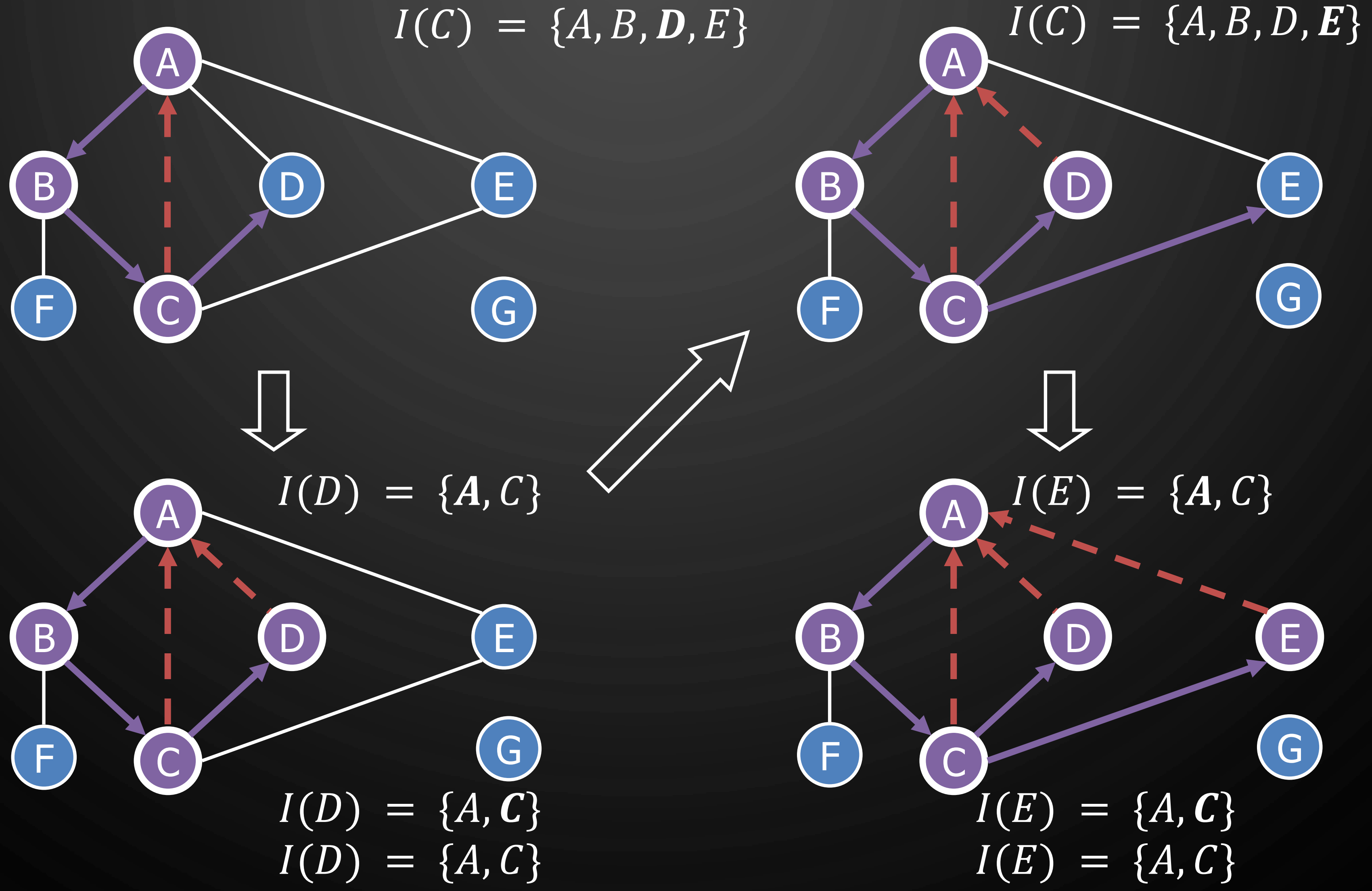
$I(C) = \{\boldsymbol{A}, B, D, E\}$



$I(C) = \{A, \boldsymbol{B}, D, E\}$

$I(C) = \{A, B, \boldsymbol{D}, E\}$

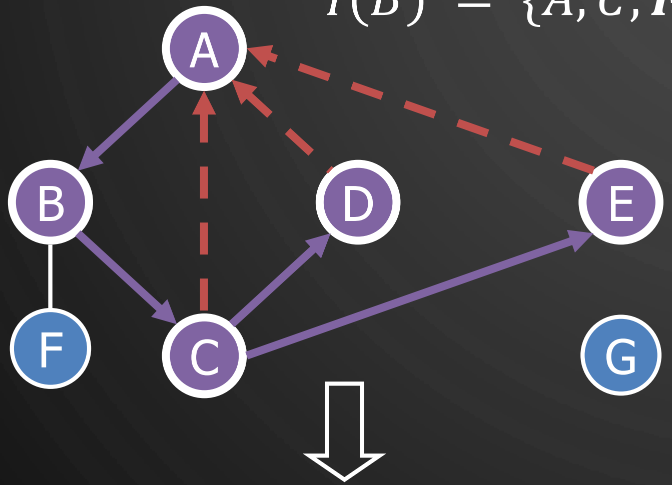
EXAMPLE


$I(C) = \{A, B, \boldsymbol{D}, E\}$
A
B
D
E
F
C
G
$I(C) = \{A, B, D, \boldsymbol{E}\}$
A
B
D
E
F
C
G
$I(D) = \{\boldsymbol{A}, C\}$
A
B
D
E
F
C
G
$I(D) = \{A, \boldsymbol{C}\}$
$I(D) = \{A, C\}$
$I(E) = \{\boldsymbol{A}, C\}$
A
B
D
E
F
C
G
$I(E) = \{A, \boldsymbol{C}\}$
$I(E) = \{A, C\}$

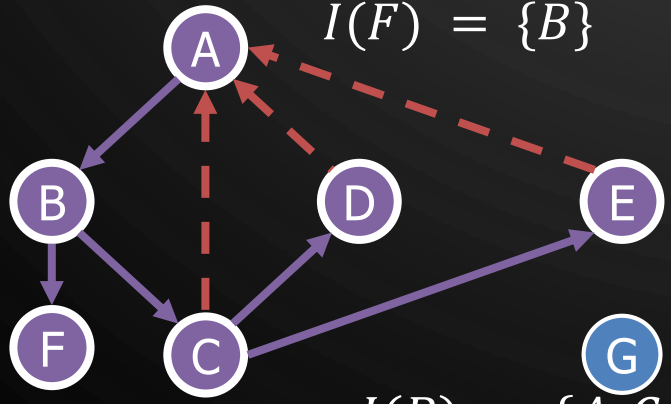
EXAMPLE

$I(C) = \{A, B, D, E\}$

$I(B) = \{A, C, \boldsymbol{F}\}$



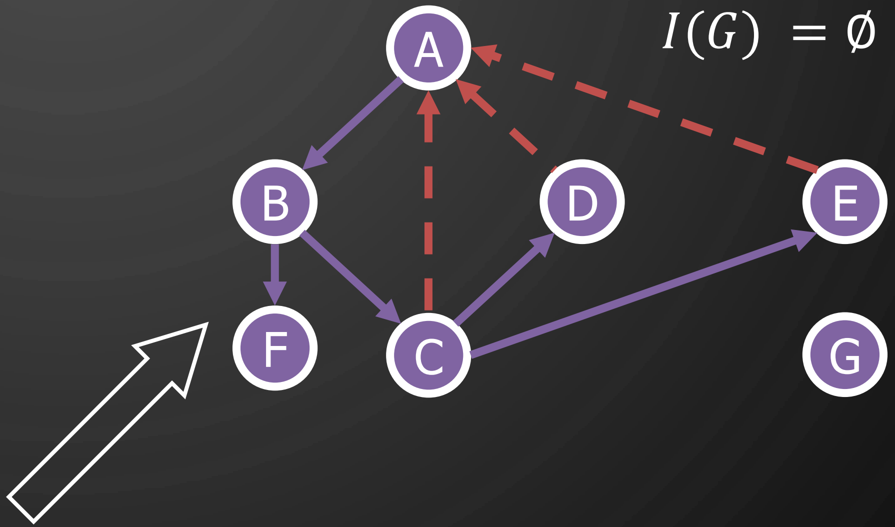
$I(F) = \{B\}$



$I(B) = \{A, C, F\}$

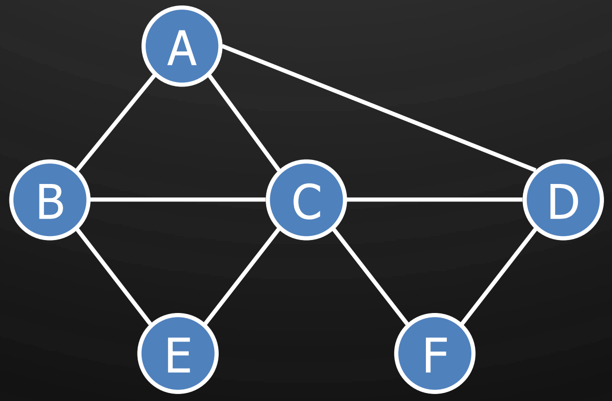
$I(A) = \{A, B, C, D\}$

$I(G) = \emptyset$

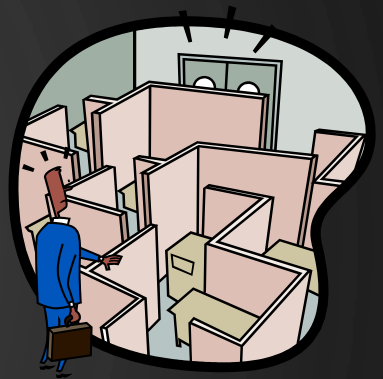


EXERCISE
DFS ALGORITHM

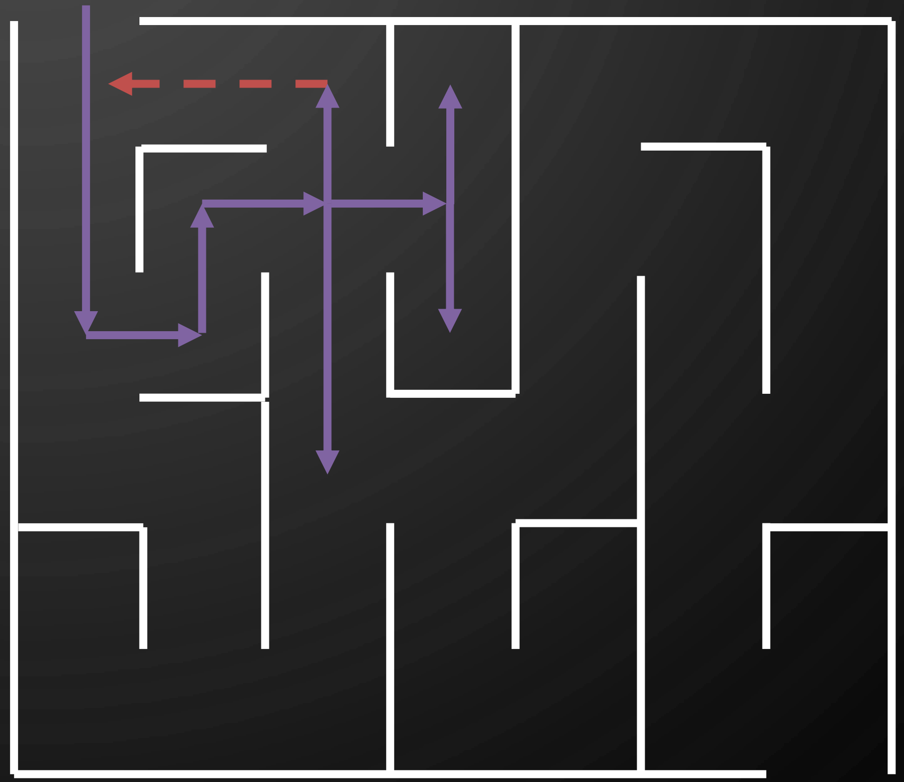
- Perform DFS of the following graph, start from vertex A
  - Assume adjacent edges are processed in alphabetical order
  - Number vertices in the order they are visited
  - Label edges as discovery or back edges



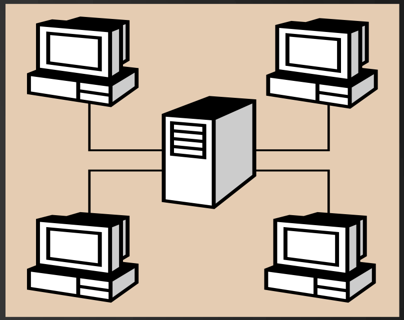
DFS AND MAZE TRAVERSAL



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



DFS ALGORITHM



- The algorithm uses a mechanism for setting and getting “labels” of vertices and edges

Algorithm DFS(G)
Input: Graph G
Output: Labeling of the edges of G as discovery edges and back edges

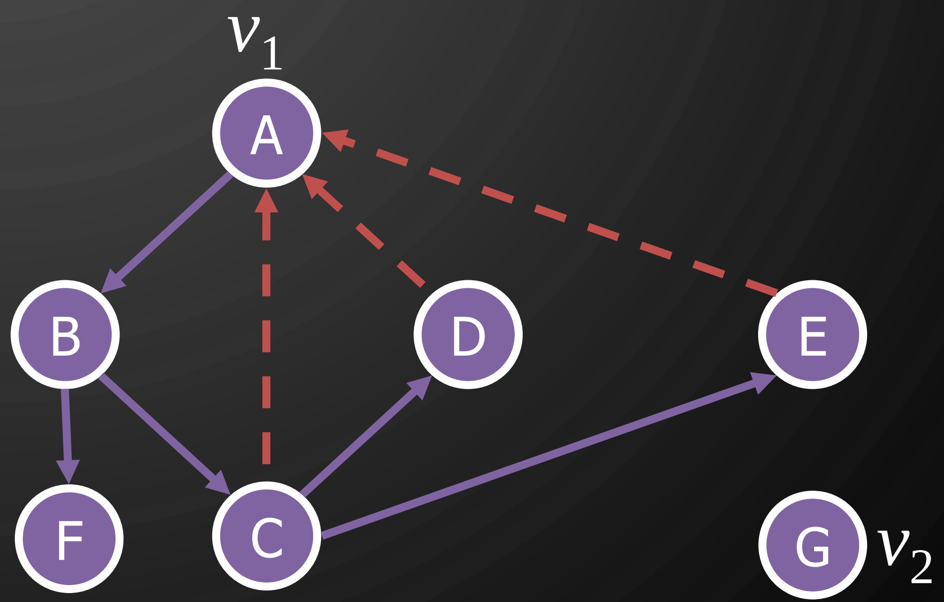
for each $v \in G$.vertices() **do**
  setLabel(v, *UNEXPLORED*)
for each $e \in G$.edges() **do**
  setLabel(e, *UNEXPLORED*)
for each $v \in G$.vertices() **do**
  if getLabel(v) = *UNEXPLORED* **then**
    DFS(G, v)

Algorithm DFS(G, v)
Input: Graph G and a start vertex v
Output: Labeling of the edges of G in the connected component of v as discovery edges and back edges

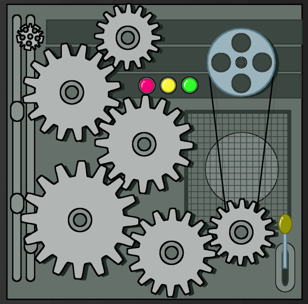
setLabel(v, *VISITED*)
for each $e \in G$.outgoingEdges(v) **do**
  if getLabel(e) = *UNEXPLORED*)
    $w \leftarrow G$.opposite(v, e)
    if getLabel(w) = *UNEXPLORED* **then**
      setLabel(e, *DISCOVERY*)
      DFS(G, w)
    else
      setLabel(e, *BACK*)

PROPERTIES OF DFS

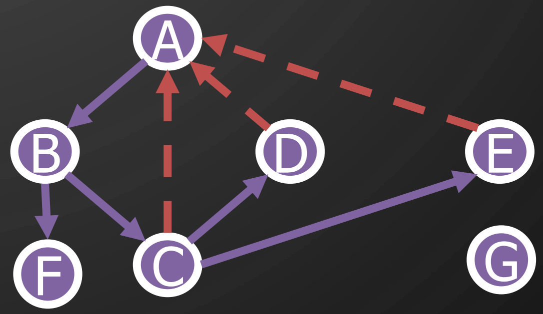
- Property 1
  - $\mathrm{DFS}(G, v)$ visits all the vertices and edges in the connected component of v
- Property 2
  - The discovery edges labeled by $\mathrm{DFS}(G, v)$ form a spanning tree of the connected component of v



ANALYSIS OF DFS



- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as $UNEXPLORED$
  - once as $VISITED$
- Each edge is labeled twice
  - once as $UNEXPLORED$
  - once as $DISCOVERY$ or $BACK$
- Function $\mathrm{DFS}(G, v)$ and the method $\mathrm{outgoingEdges}(\,)$ are called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$



APPLICATION

PATH FINDING

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call $\mathrm{DFS}(G, u)$ with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G,v,z)
Input: Graph G, a start vertex v,
       a goal vertex z
Output: Path between v and z
setLabel(v, VISITED)
S.push(v)
if v = z then
  return S.elements()
for each e ∈ G.outgoingEdges(v) do
  if getLabel(e) = UNEXPLORED) then
    w ← G.opposite(v,e)
    if getLabel(w) = UNEXPLORED then
      setLabel(e, DISCOVERY)
      S.push(e)
      pathDFS(G,w)
      S.pop()
    else
      setLabel(e, BACK)
S.pop()
```

APPLICATION

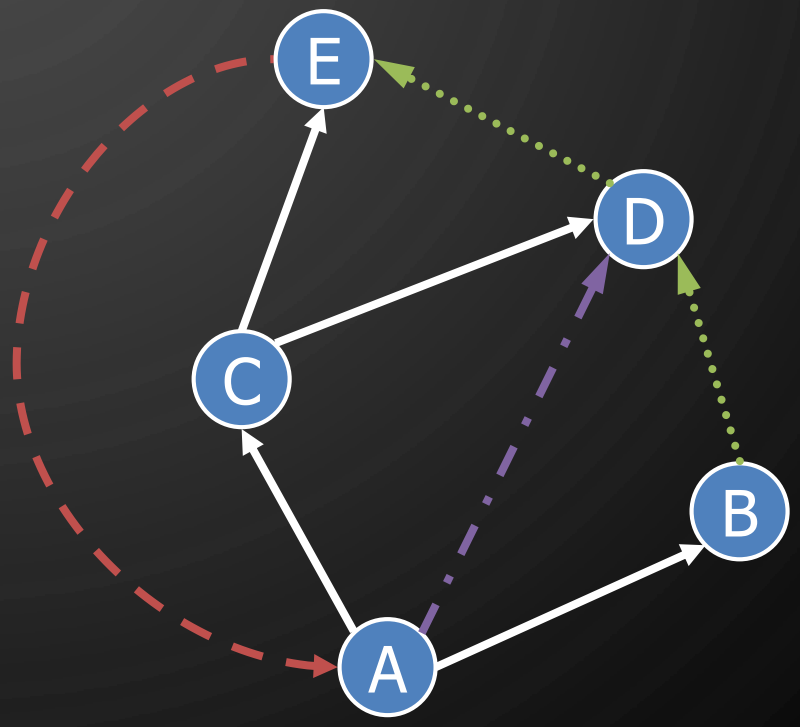
CYCLE FINDING

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex w

```
Algorithm cycleDFS(G,v)
Input: Graph G, a start vertex v
Output: Cycle containing v
setLabel(v,VISITED)
S.push(v)
for each e ∈ G.outgoingEdges(v) do
  if getLabel(e) = UNEXPLORED) then
    w ← G.opposite(v,e)
    S.push(e)
    if getLabel(w) = UNEXPLORED then
      setLabel(e,DISCOVERY)
      cycleDFS(G,w)
      S.pop()
    else
      Stack T ← ∅
      repeat
        T.push(S.pop())
      until T.top() = w
      return T.elements()
S.pop()
```

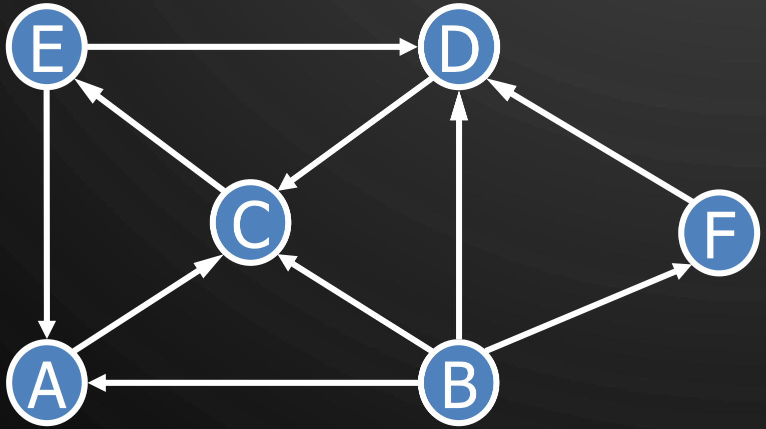
DIRECTED DFS

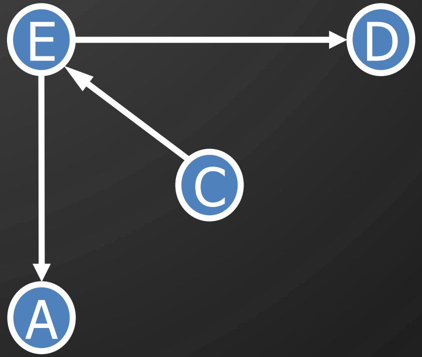
- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting at a vertex s determines the vertices reachable from s

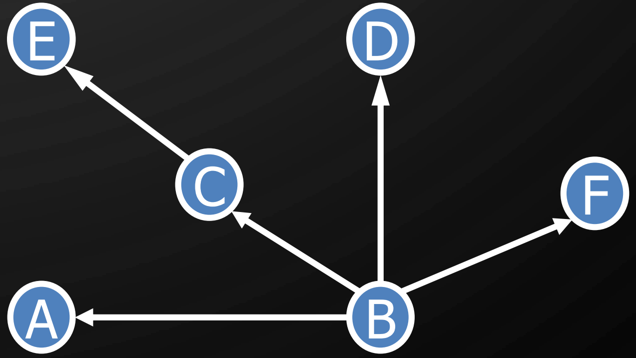


REACHABILITY

- DFS tree rooted at v: vertices reachable from v via directed paths

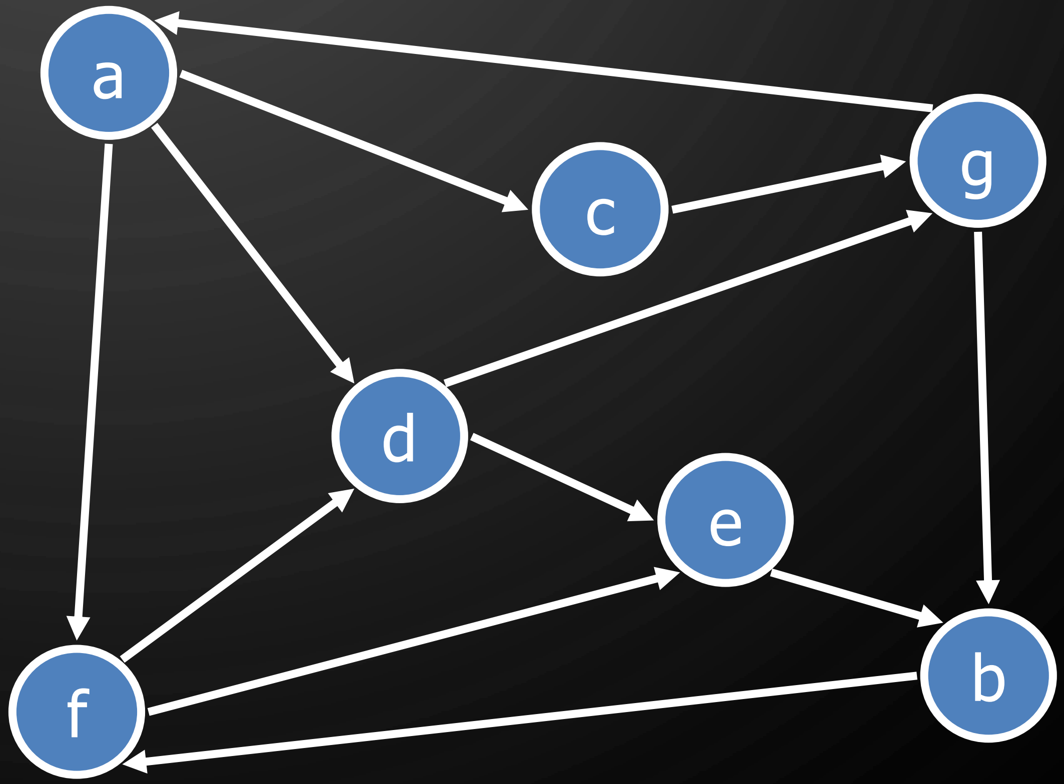




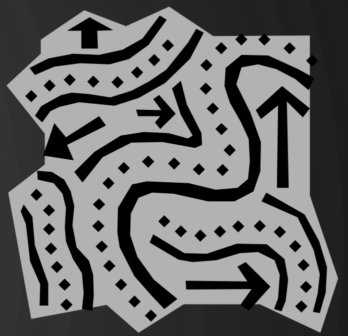


STRONG CONNECTIVITY

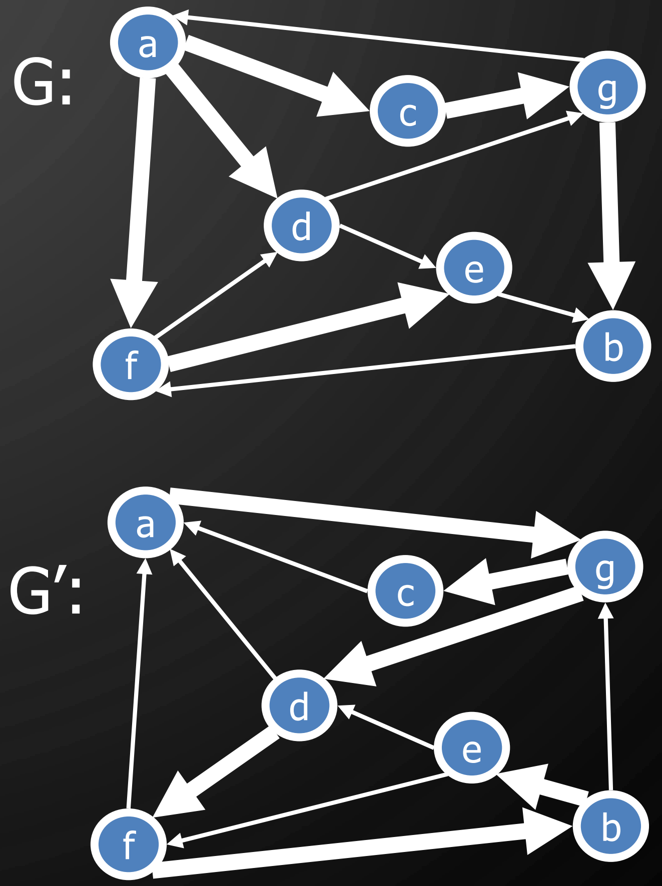
- Each vertex can reach all other vertices



STRONG CONNECTIVITY ALGORITHM

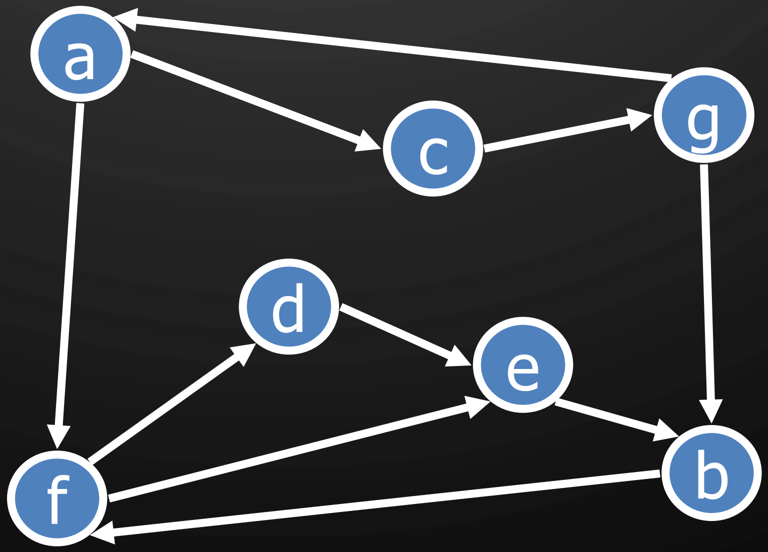


- Pick a vertex v in G
- Perform a DFS from v in G
  - If there's a w not visited, print "no"
- Let G' be G with edges reversed
- Perform a DFS from v in G'
  - If there's a w not visited, print "no"
  - Else, print "yes"
- Running time: $O(n + m)$



STRONGLY CONNECTED COMPONENTS

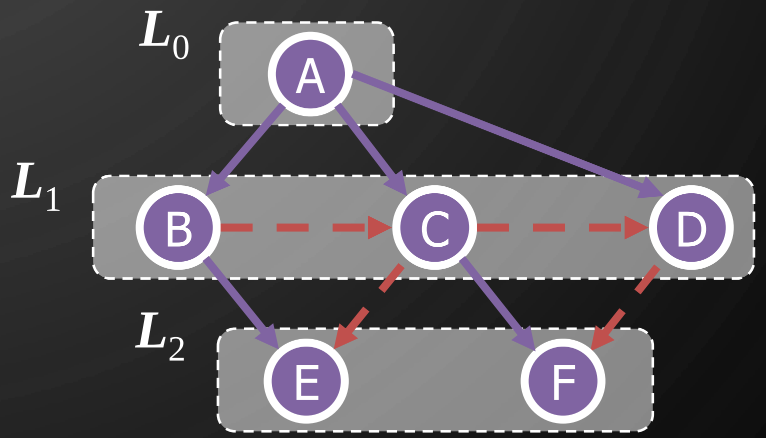
- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in $O(n + m)$ time using DFS, but is more complicated (similar to biconnectivity).



$\{a, c, g\}$

$\{f, d, e, b\}$

BREADTH-FIRST SEARCH



BREADTH-FIRST SEARCH

- **Breadth-first search (BFS)** is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with n vertices and m edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

BFS ALGORITHM

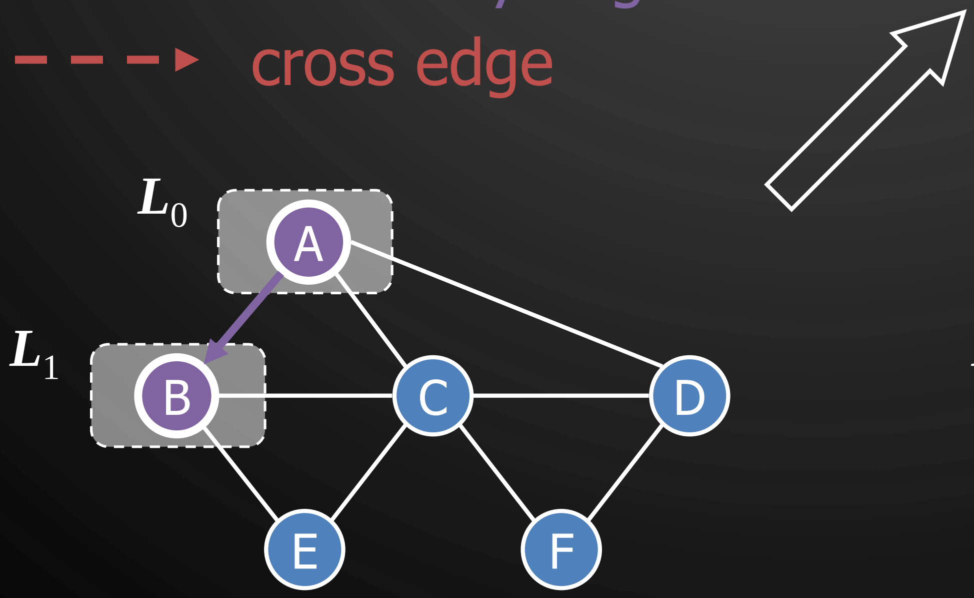
- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

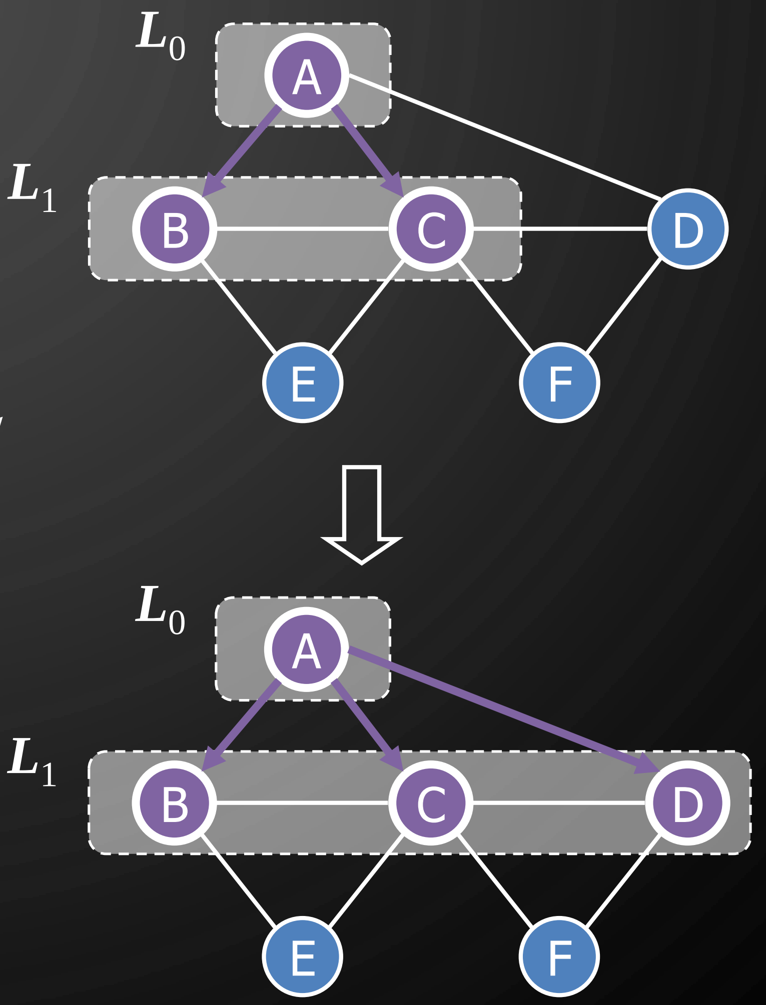
```
Algorithm BFS(G)
Input: Graph G
Output: Labeling of the edges and
   partition of the vertices of G
for each v ∈ G.vertices() do
  setLabel(v, UNEXPLORED)
for each e ∈ G.edges() do
  setLabel(e, UNEXPLORED)
for each v ∈ G.vertices() do
  if getLabel(v) = UNEXPLORED then
    BFS(G, v)
```

```
Algorithm BFS(G, s)
Input: Graph G, a start vertex s
List L0 ← {s}
setLabel(s, VISITED)
i ← 0
while ¬Li.isEmpty() do
  List Li+1 ← ∅
  for each v ∈ Li do
    for each e ∈ G.outgoingEdges(v) do
      if getLabel(e) = UNEXPLORED then
        w ← G.opposite(v, e)
        if getLabel(w) = UNEXPLORED then
          setLabel(e, DISCOVERY)
          setLabel(w, VISITED)
          Li+1 ← Li+1 ∪ {w}
        else
          setLabel(e, CROSS)
  i ← i + 1
```

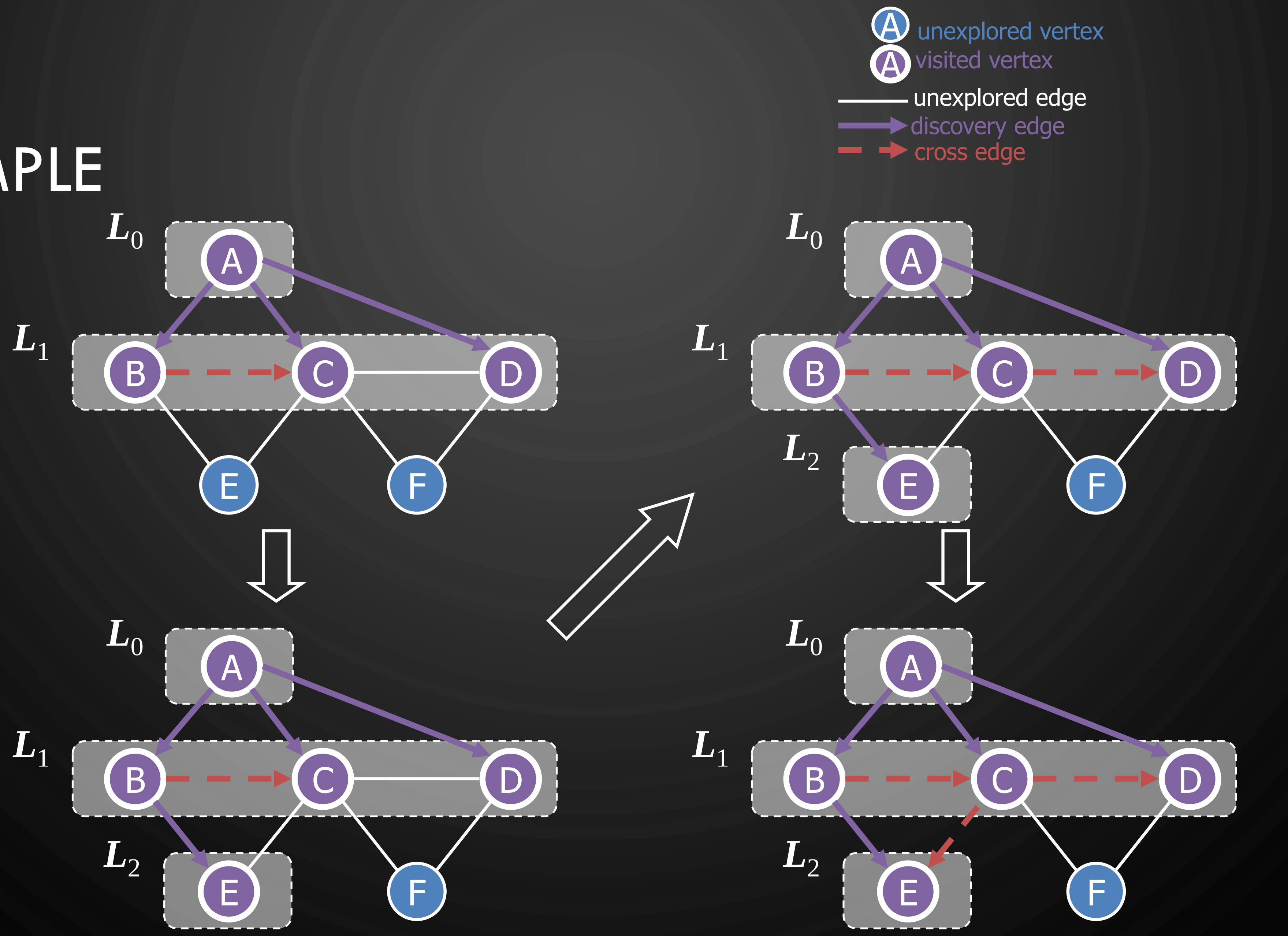
EXAMPLE

- unexplored vertex
- visited vertex
- unexplored edge
- discovery edge
- cross edge



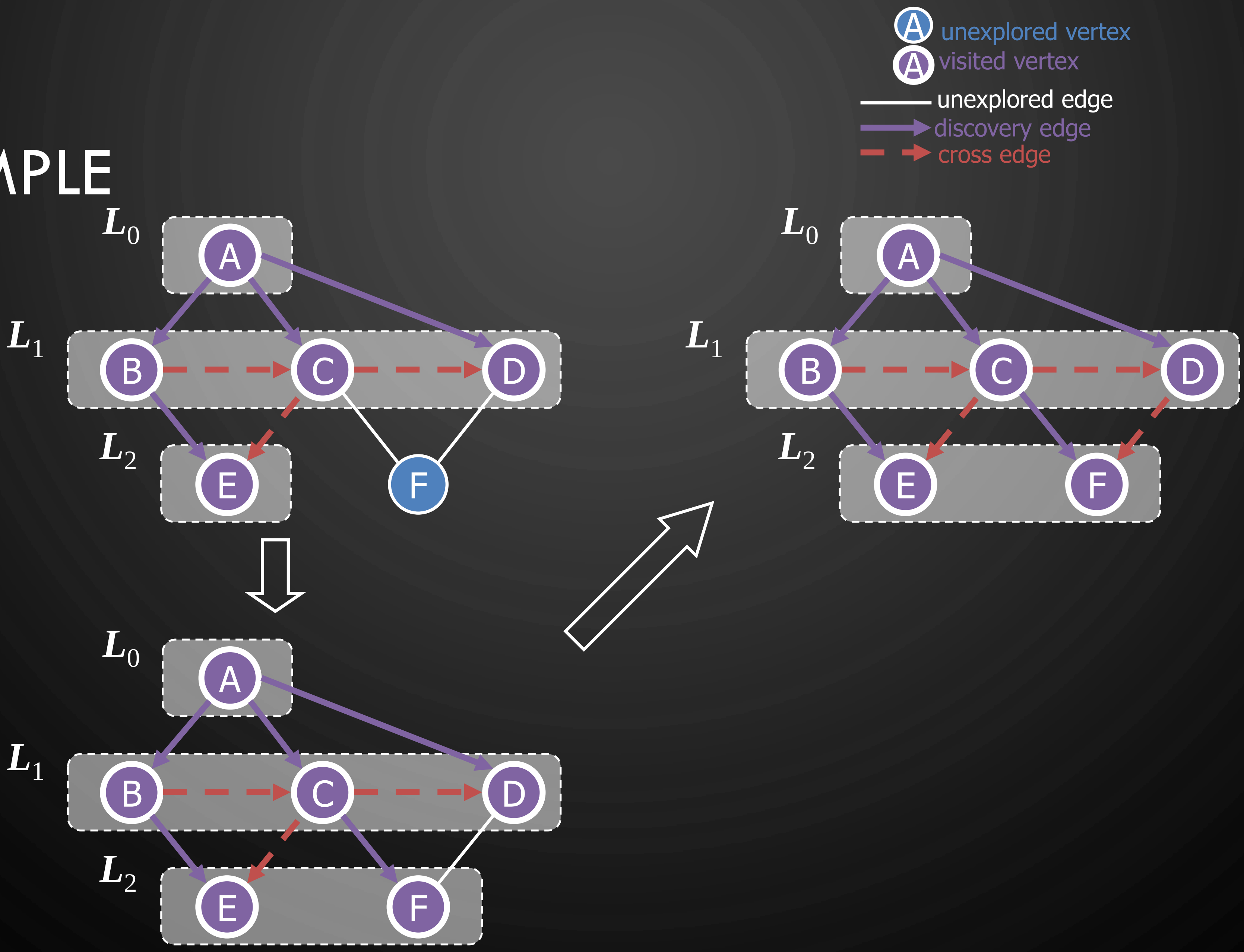


EXAMPLE



A unexplored vertex
A visited vertex
unexplored edge
discovery edge
cross edge
L_0
L_1
L_2
A
B
C
D
E
F

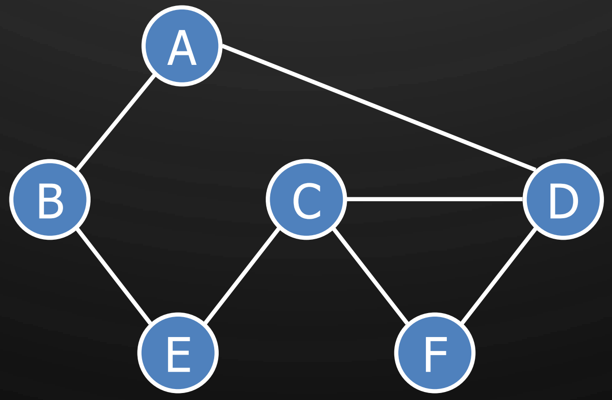
EXAMPLE



A
unexplored vertex
A
visited vertex
unexplored edge
discovery edge
cross edge
L_0
L_1
L_2
A
B
C
D
E
F

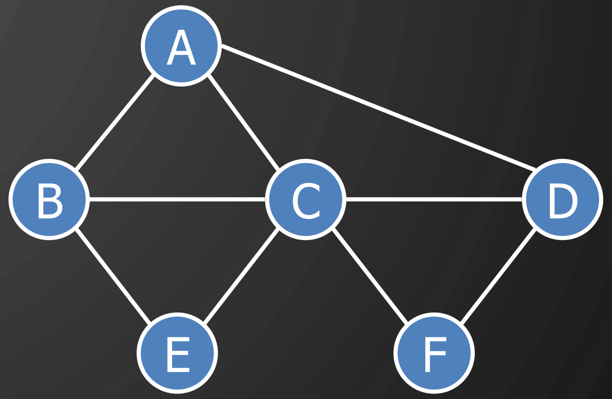
EXERCISE
BFS ALGORITHM

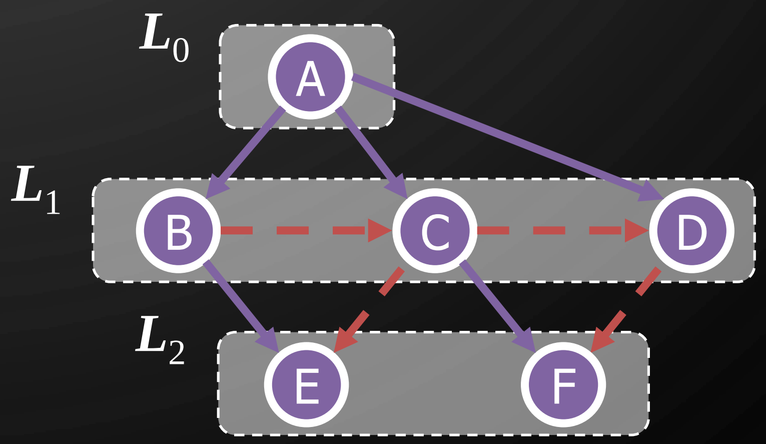
- Perform BFS of the following graph, start from vertex F
  - Assume adjacent edges are processed in alphabetical order
  - Number vertices in the order they are visited and note the level they are in
  - Label edges as discovery or cross edges



PROPERTIES

- Notation
  - G_s: connected component of s
- Property 1
  - $\mathrm{BFS}(G, s)$ visits all the vertices and edges of G_s
- Property 2
  - The discovery edges labeled by $\mathrm{BFS}(G, s)$ form a spanning tree T_s of G_s
- Property 3
  - For each vertex $v \in L_i$
    - The path of T_s from s to v has i edges
    - Every path from s to v in G_s has at least i edges





ANALYSIS

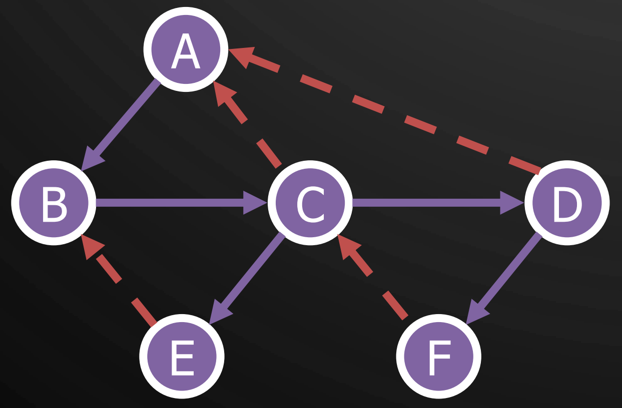
- Setting/getting a vertex/edge label takes O(1) time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method $\mathrm{outgoingEdges}(\,)$ is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

APPLICATIONS

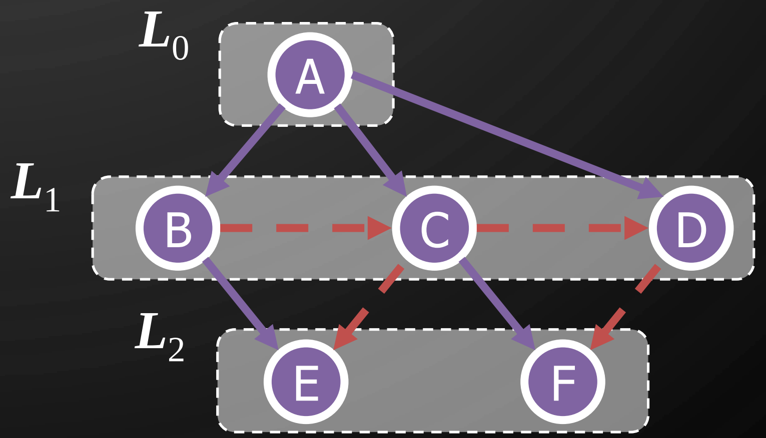
- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n+m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

DFS VS. BFS

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS

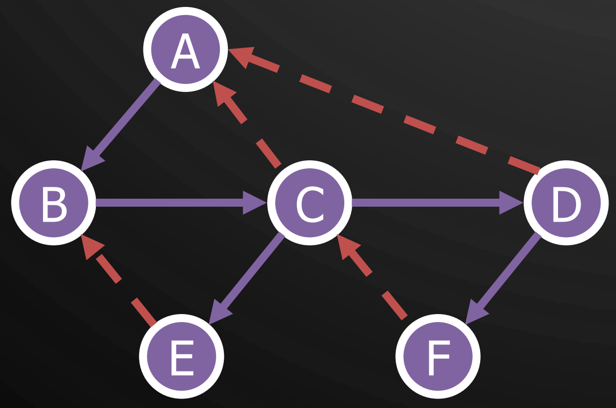


BFS

DFS VS. BFS

Back edge (v, w)

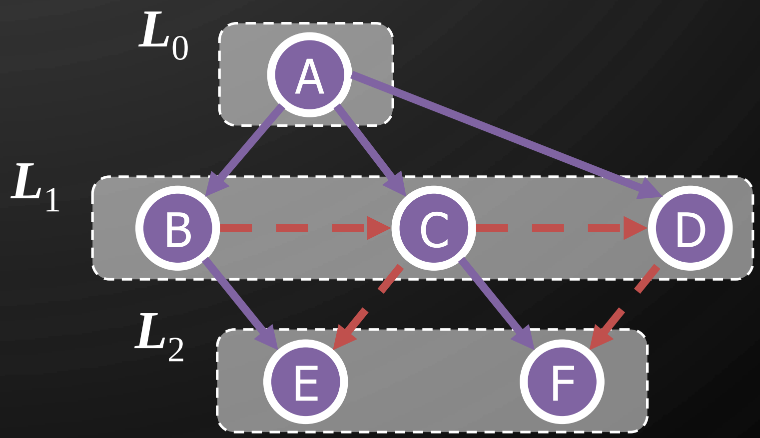
- w is an ancestor of v in the tree of discovery edges



DFS

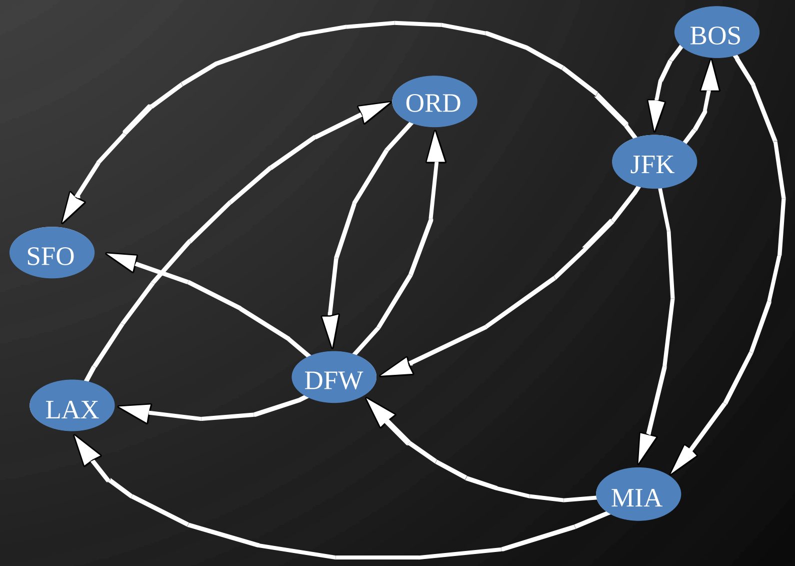
Cross edge (v, w)

- w is in the same level as v or in the next level in the tree of discovery edges



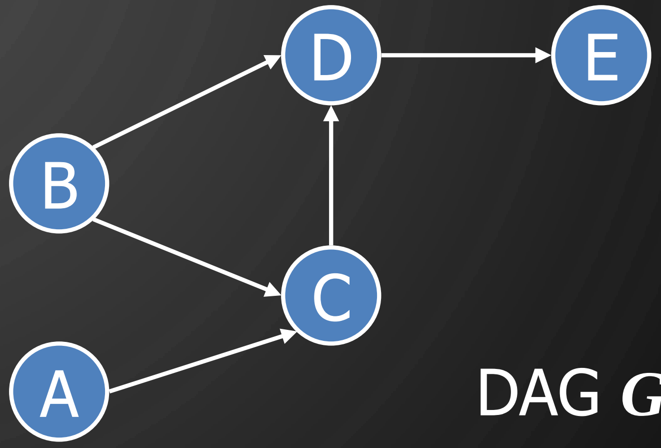
BFS

TOPOLOGICAL ORDERING

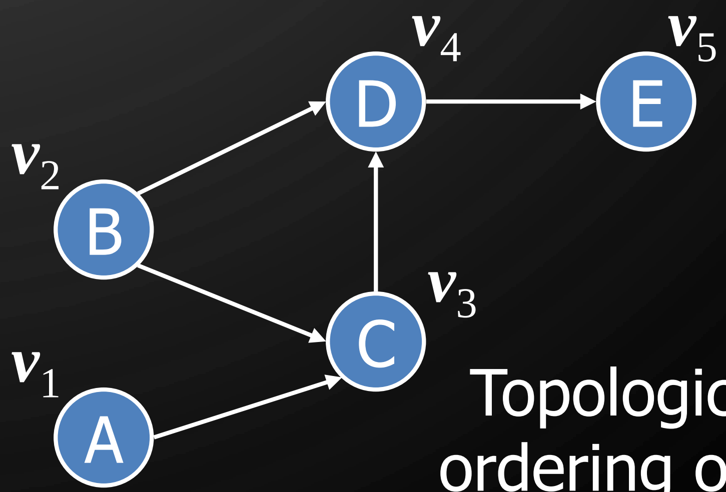


DAGS AND TOPOLOGICAL ORDERING

- A **directed acyclic graph (DAG)** is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering
  - $v_1, \dots, v_n$
  - Of the vertices such that for every edge (v_i, v_j), we have $i < j$
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints
- Theorem - A digraph admits a topological ordering if and only if it is a DAG



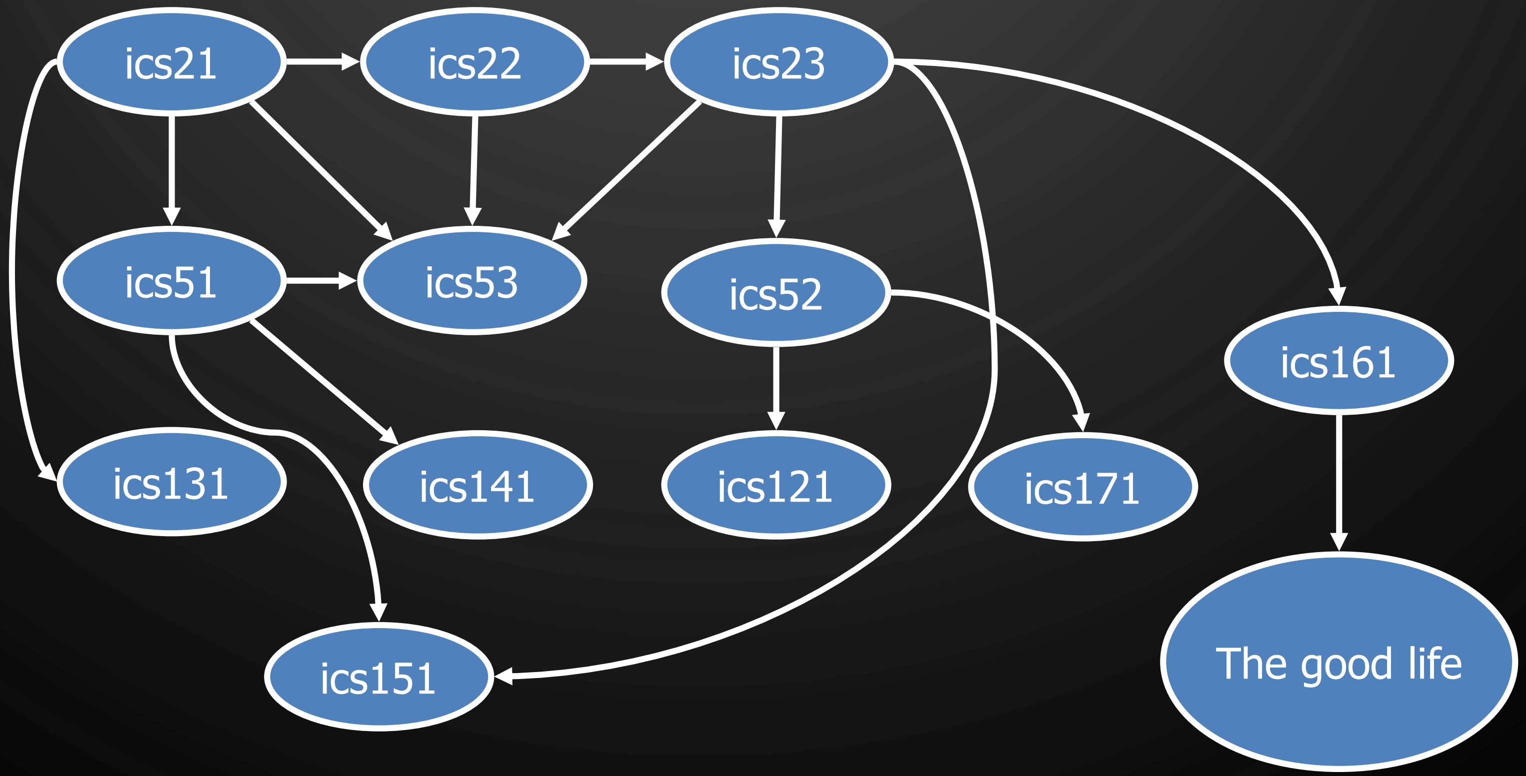
DAG G



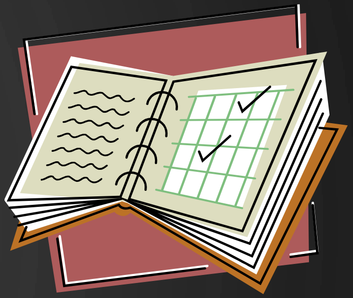
Topological ordering of G

APPLICATION

- Scheduling: edge (a, b) means task a must be completed before b can be started

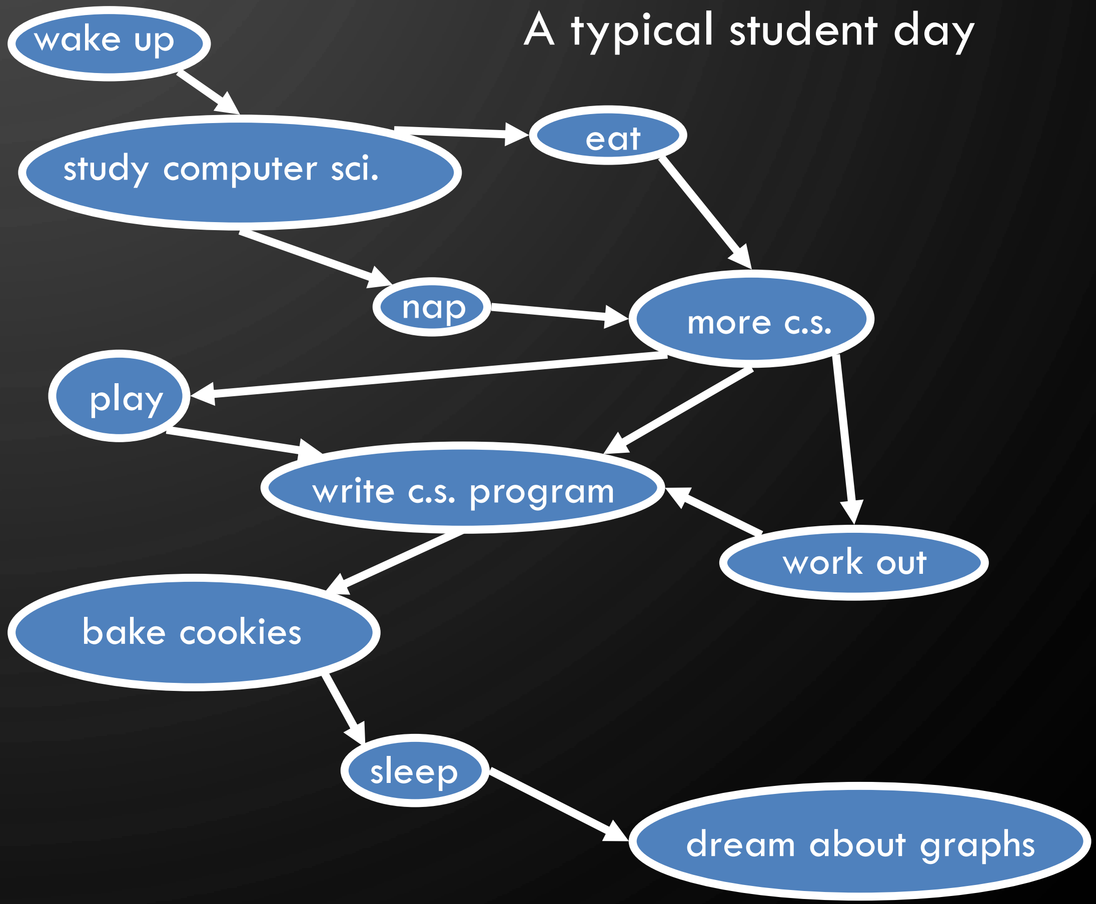


EXERCISE
TOPOLOGICAL SORTING



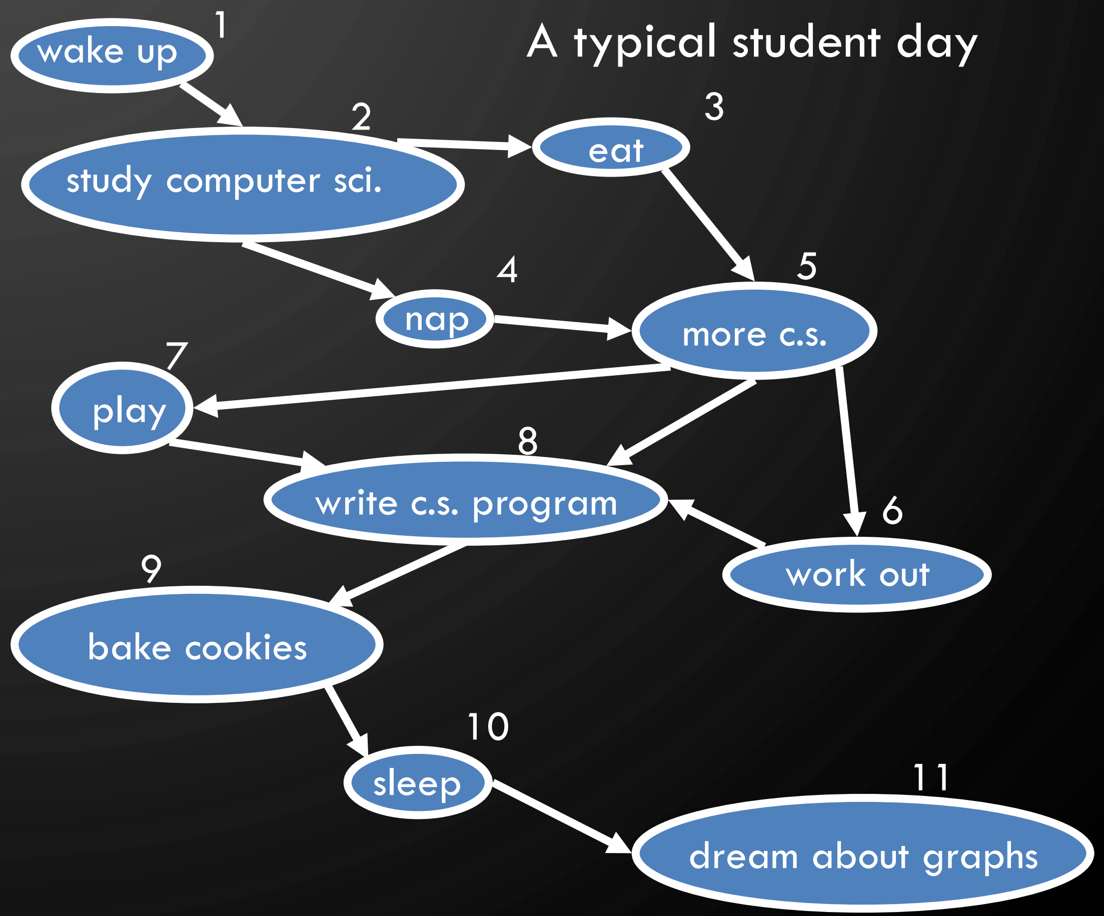
- Number vertices, so that (u, v) in E implies $u < v$

A typical student day



EXERCISE
TOPOLOGICAL SORTING

- Number vertices, so that (u, v) in E implies $u < v$



ALGORITHM FOR TOPOLOGICAL SORTING

```
Algorithm TopologicalSort(G)
Input: Directed Acyclic Graph (DAG) G
Output: Topological ordering of G
1.  H ← G
2.  n ← G.numVertices()
3.  while ¬H.isEmpty() do
4.     Let v be a vertex with no outgoing edges
5.     Label v ← n
6.     n ← n − 1
7.     H.removeVertex(v)
```

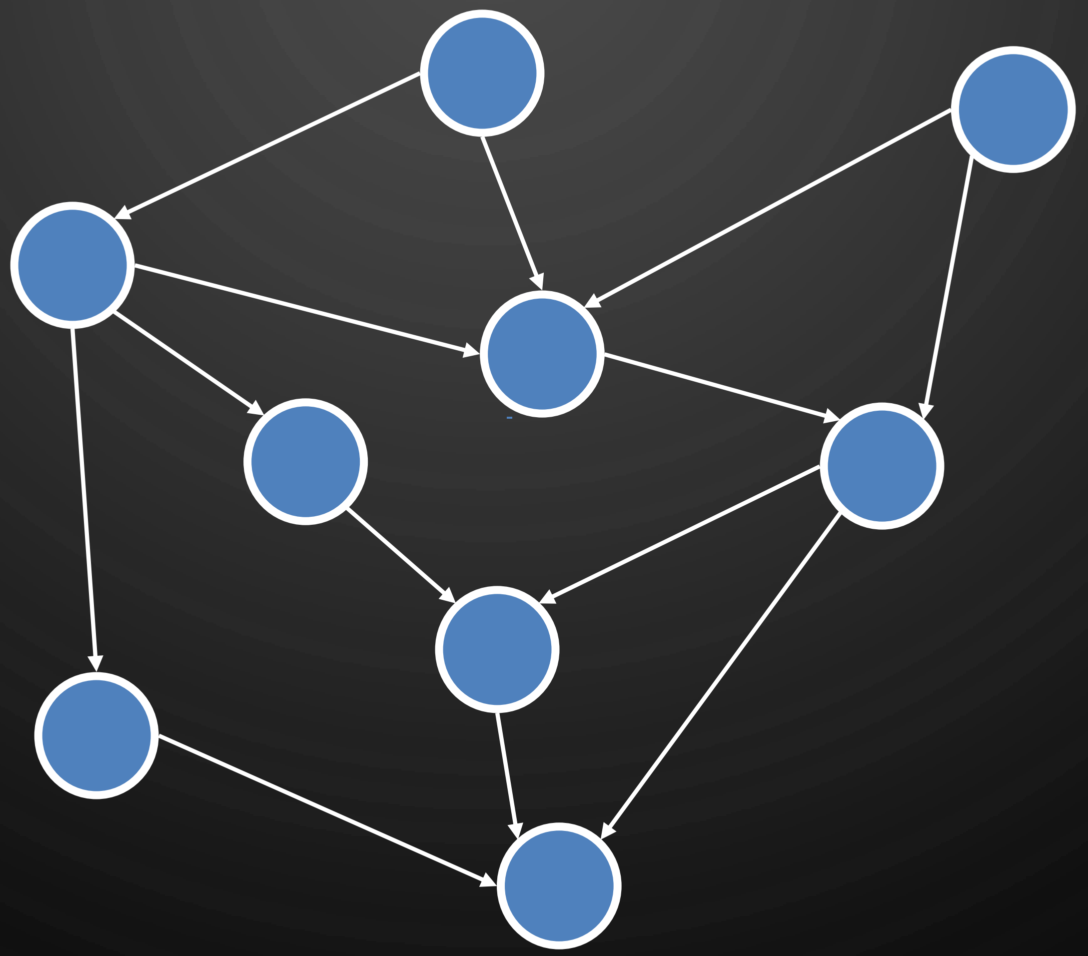
IMPLEMENTATION WITH DFS

- Simulate the algorithm by using depth-first search
- $O(n+m)$ time.

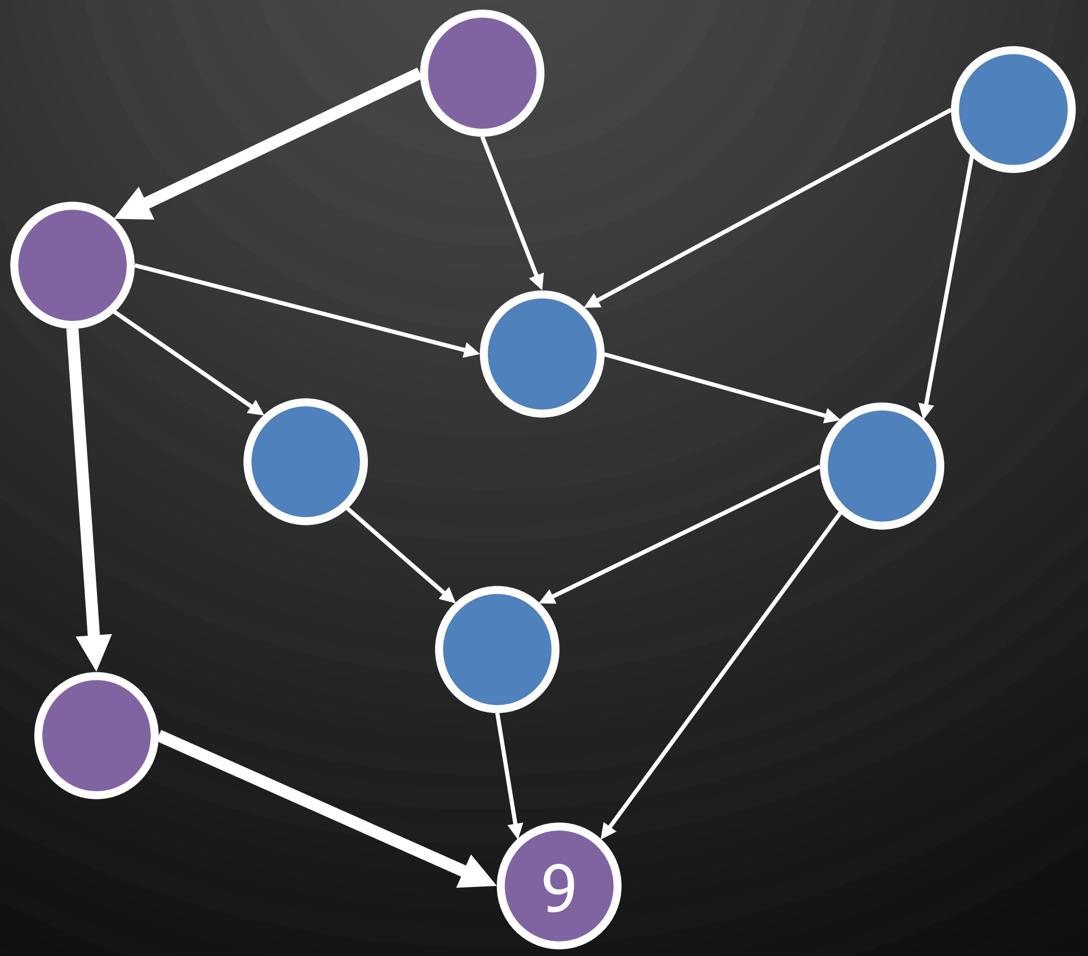
Algorithm topologicalDFS(G)
Input: DAG G
Output: Topological ordering of G
1. $n \leftarrow G$.numVertices()
2. Initialize all vertices as *UNEXPLORED*
3. **for each** vertex $v \in G$.vertices() **do**
4.   **if** getLabel(v) = *UNEXPLORED* **then**
5.   topologicalDFS(G, v)

Algorithm topologicalDFS(G, v)
Input: DAG G, start vertex v
Output: Labeling of the vertices of G in the connected component of v
1. setLabel(v, *VISITED*)
2. **for each** $e \in G$.outgoingEdges(v) **do**
3. $w \leftarrow G$.opposite(v, e)
4.   **if** getLabel(w) = *UNEXPLORED* **then**
5.   //e is a discovery edge
6.   topologicalDFS(G, w)
7.   **else**
8.   //e is a forward, cross, or back edge
9. Label v with topological number n
10. $n \leftarrow n - 1$

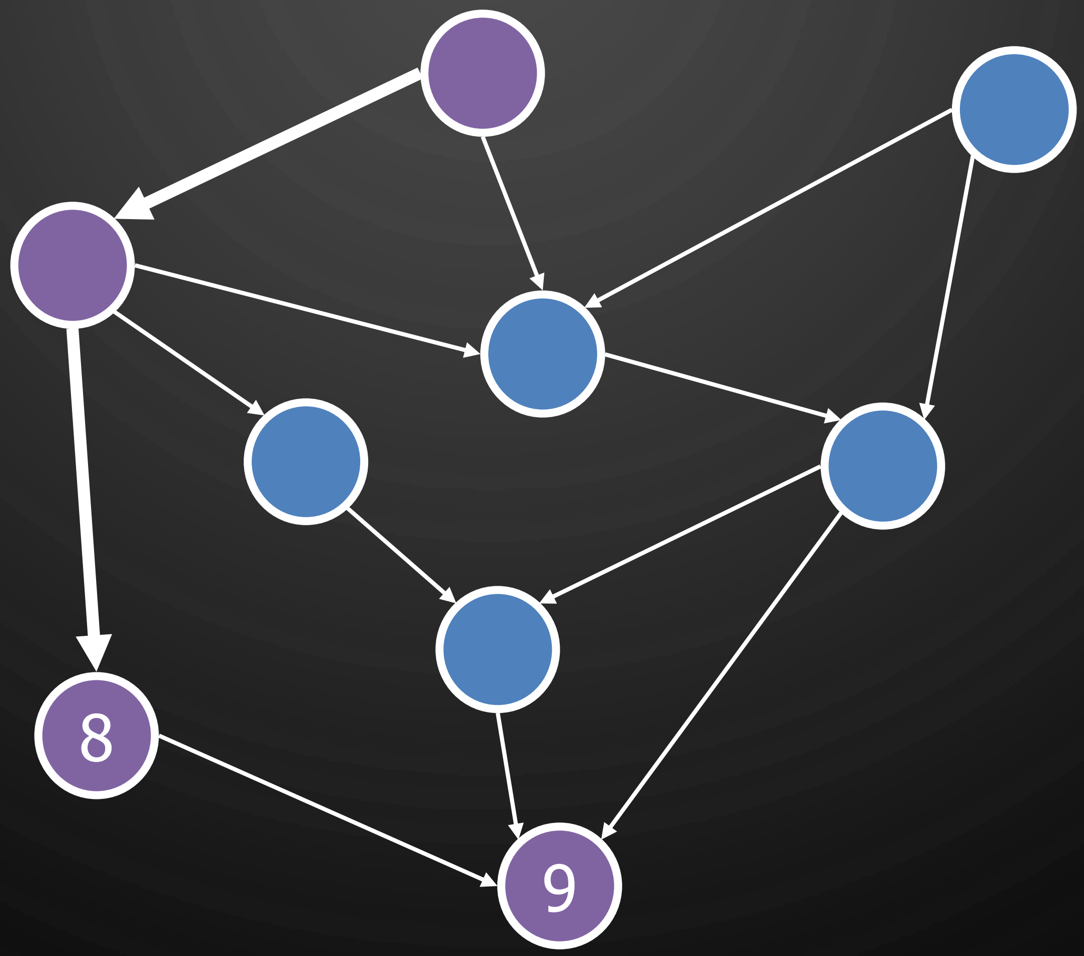
TOPOLOGICAL SORTING EXAMPLE



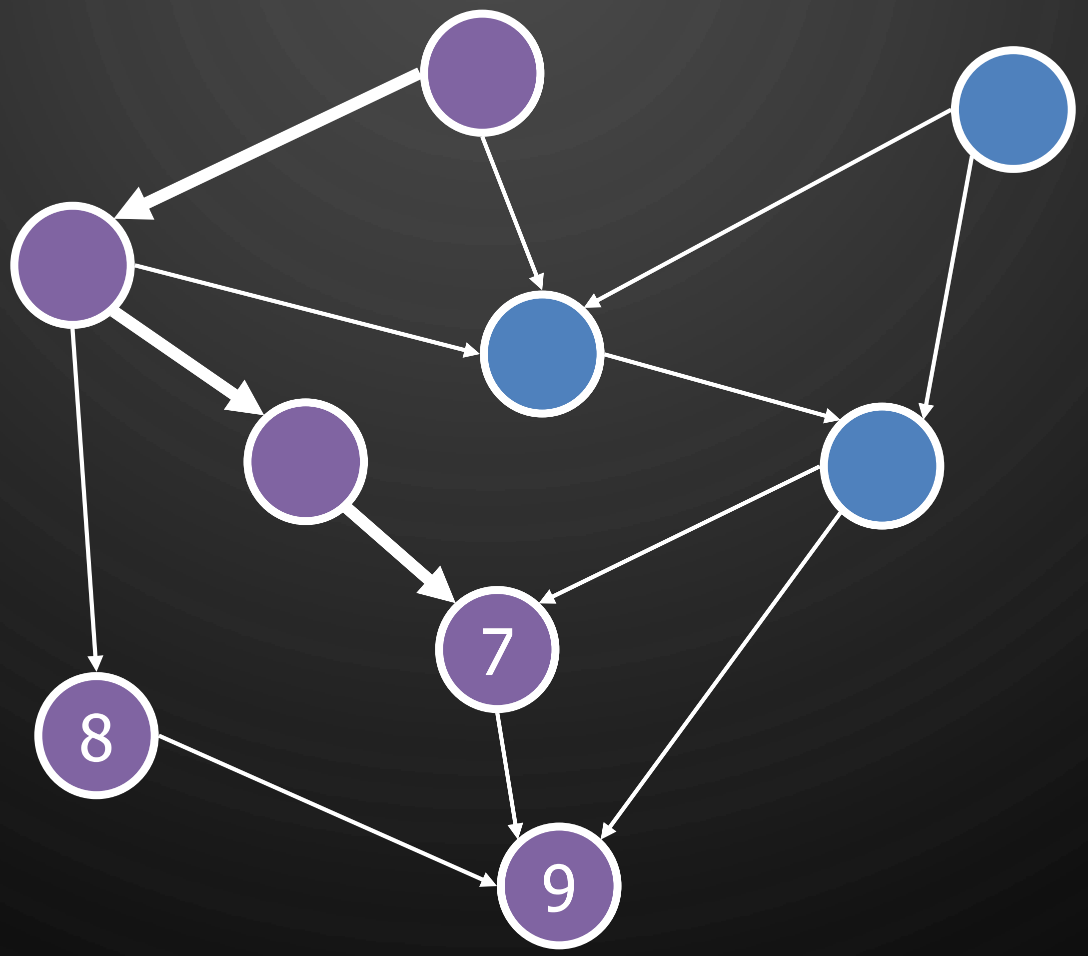
TOPOLOGICAL SORTING EXAMPLE



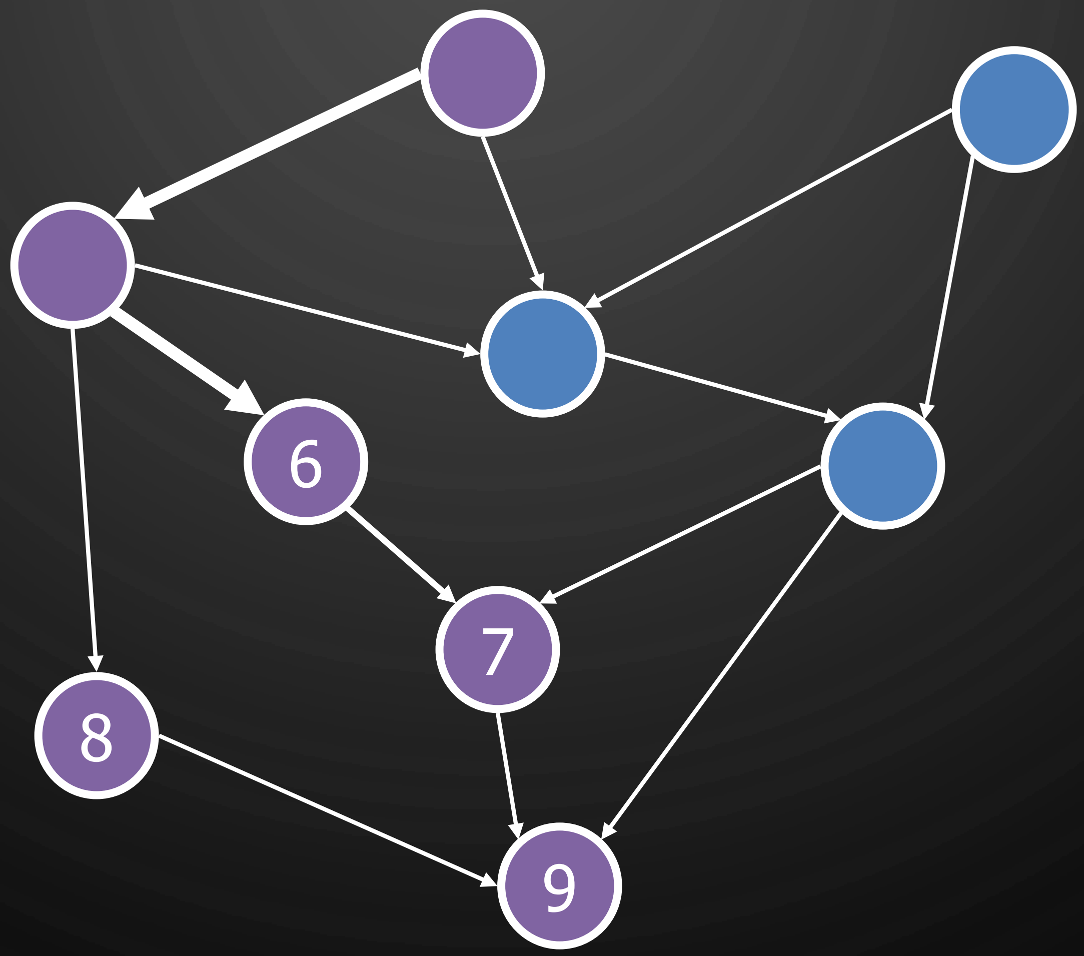
TOPOLOGICAL SORTING EXAMPLE



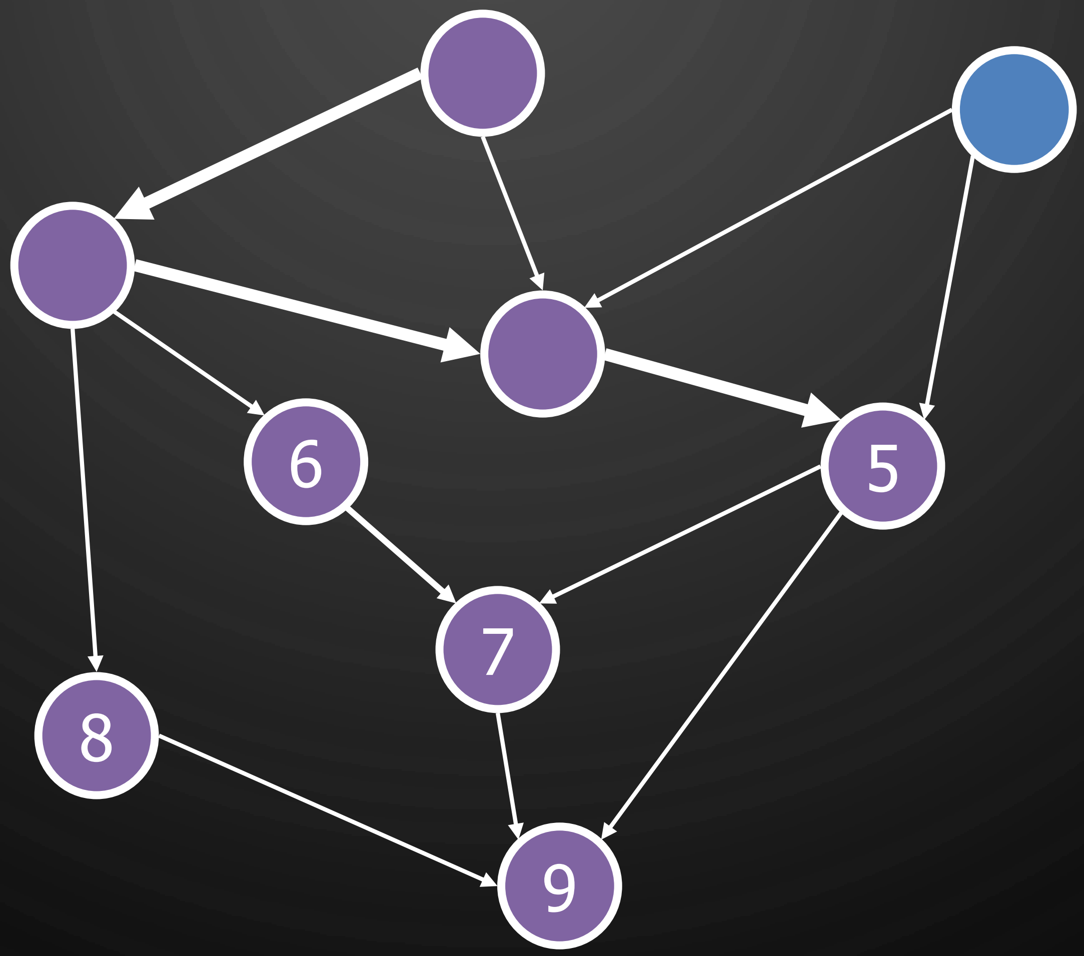
TOPOLOGICAL SORTING EXAMPLE



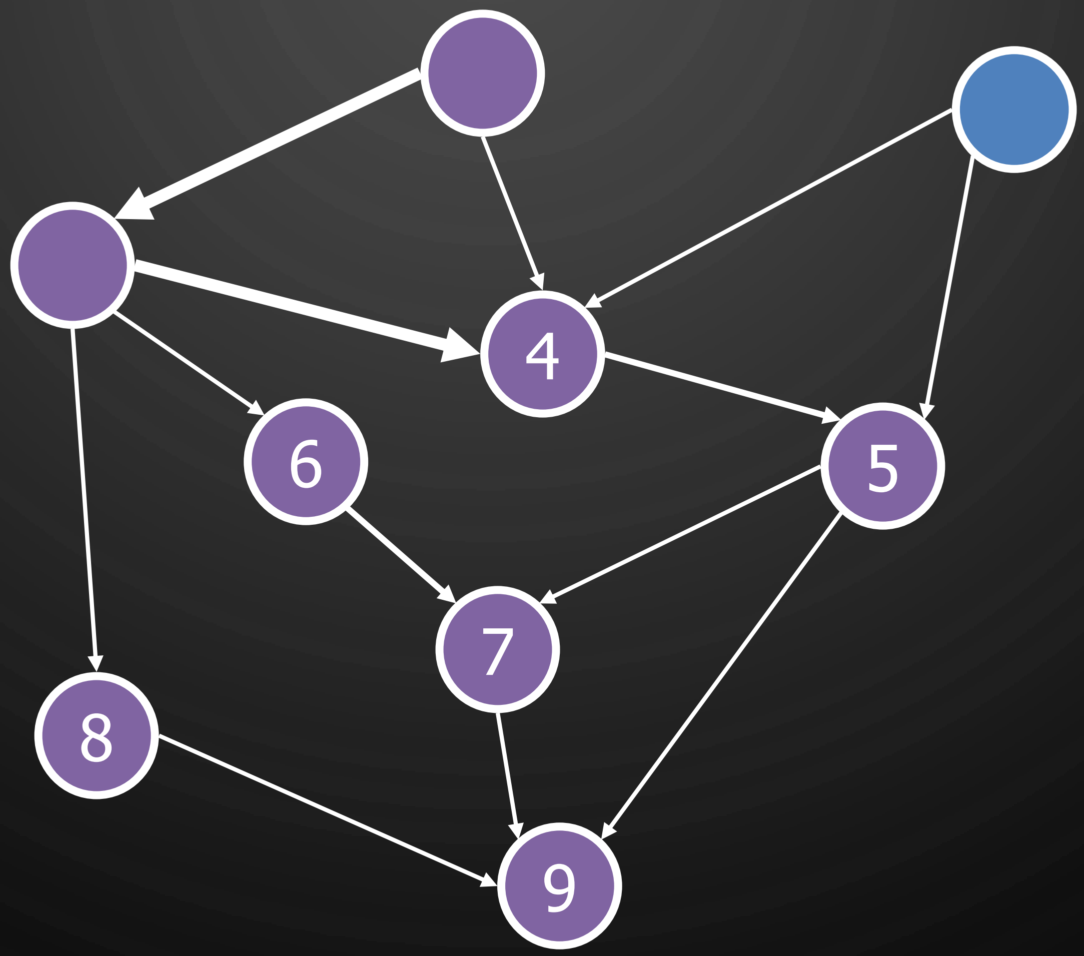
TOPOLOGICAL SORTING EXAMPLE



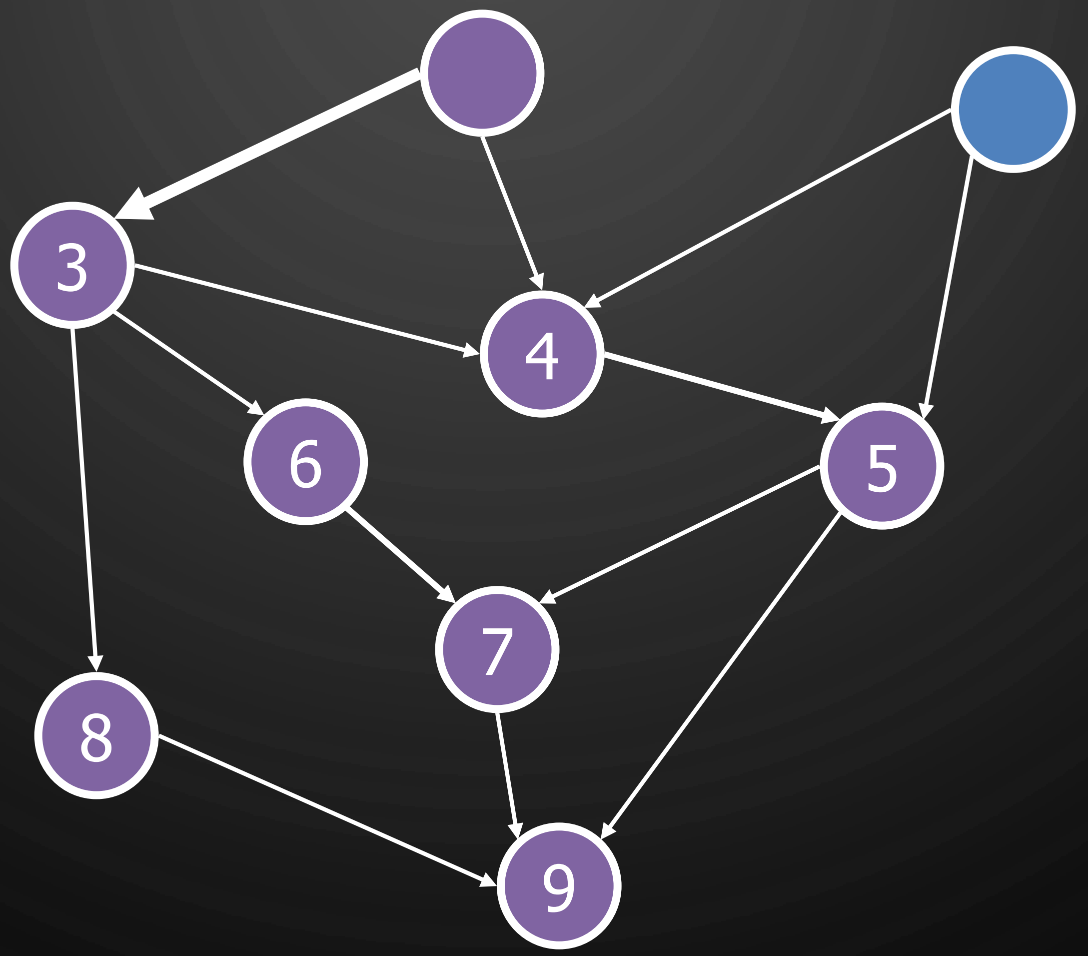
TOPOLOGICAL SORTING EXAMPLE



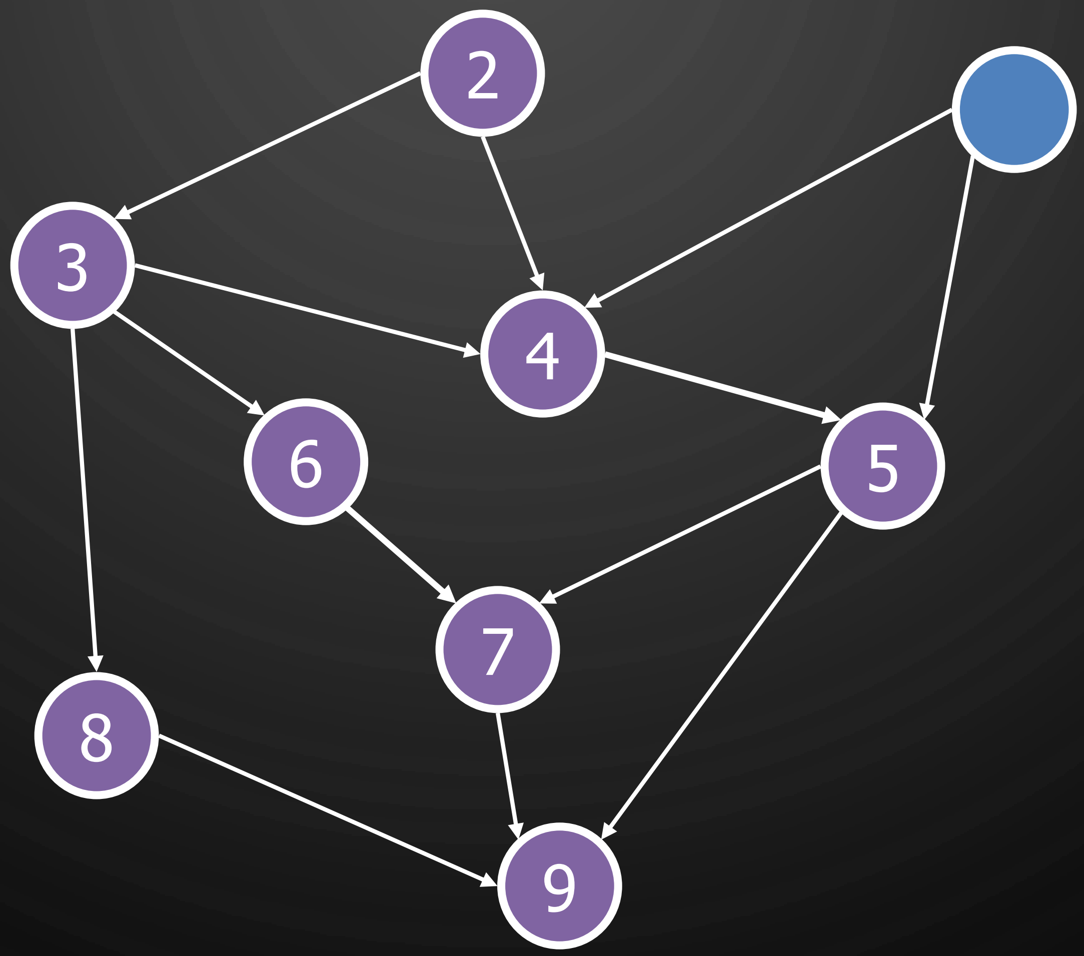
TOPOLOGICAL SORTING EXAMPLE



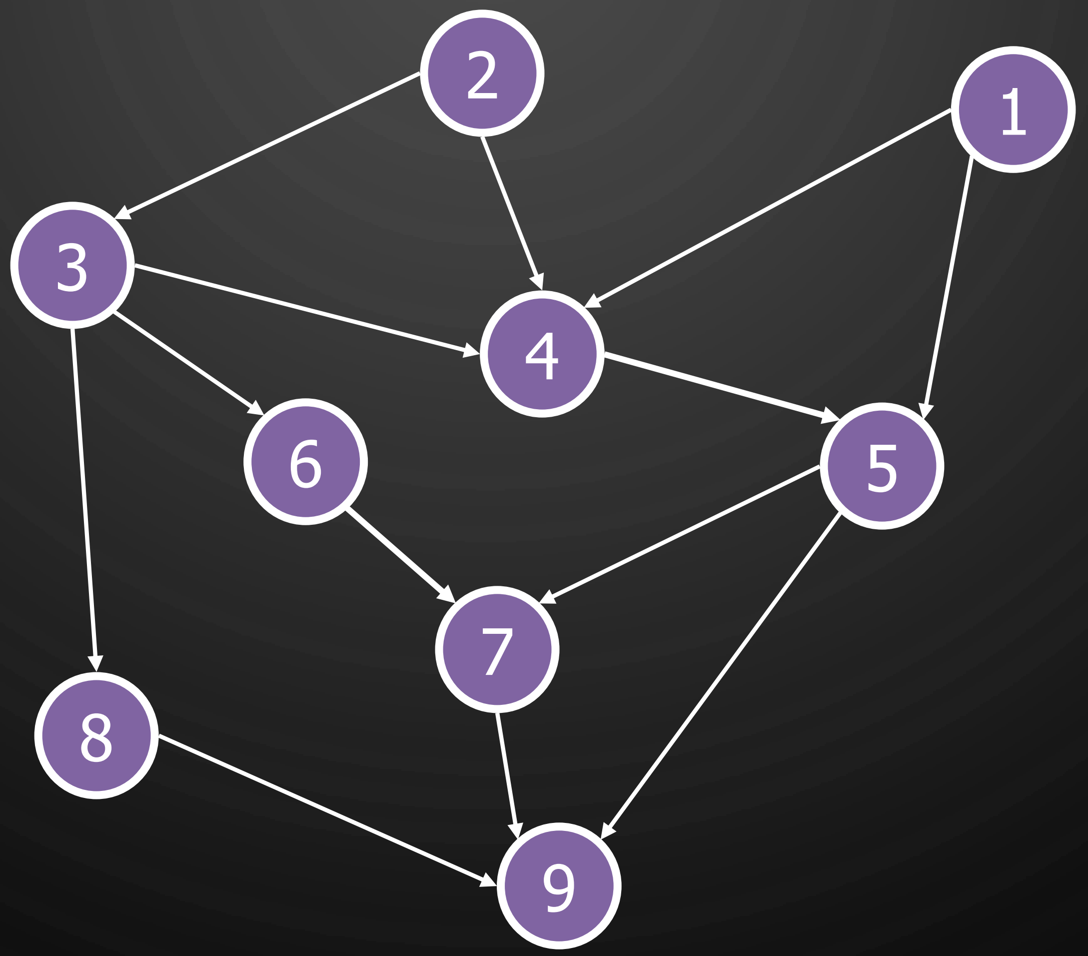
TOPOLOGICAL SORTING EXAMPLE



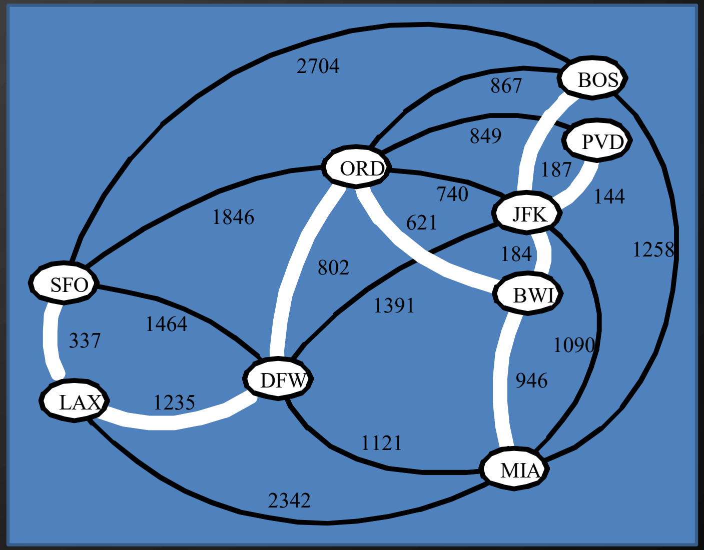
TOPOLOGICAL SORTING EXAMPLE



TOPOLOGICAL SORTING EXAMPLE

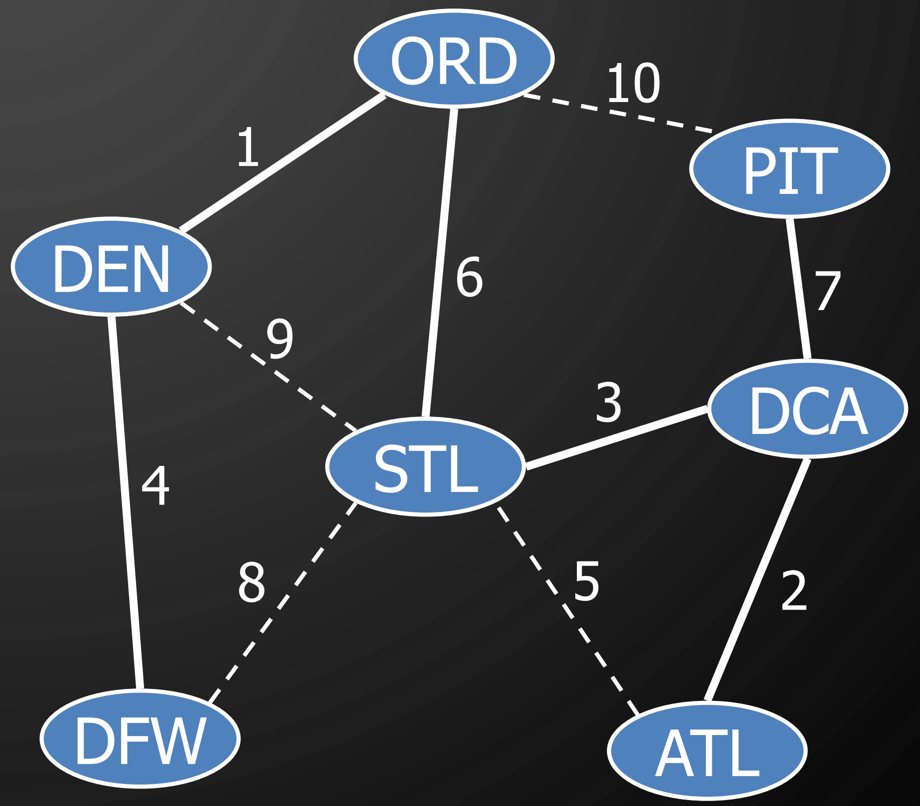


MINIMUM SPANNING TREES



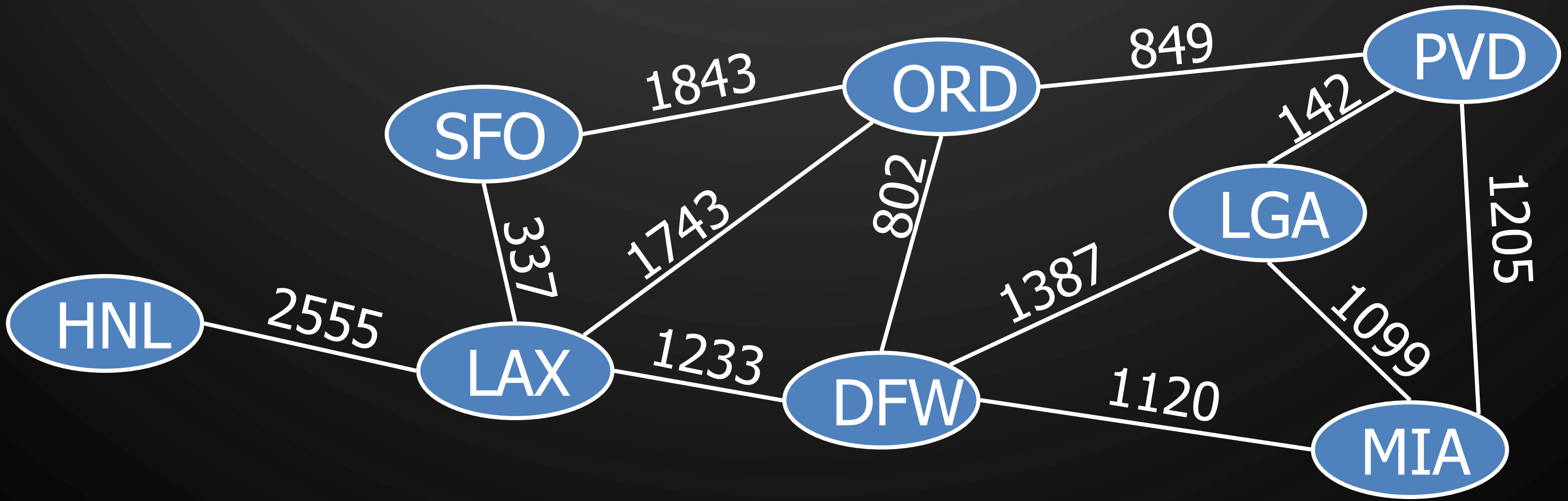
MINIMUM SPANNING TREE

- Minimum spanning tree (MST)
  - Spanning tree of a weighted graph with minimum total edge weight
- Applications
  - Communications networks
  - Transportation networks



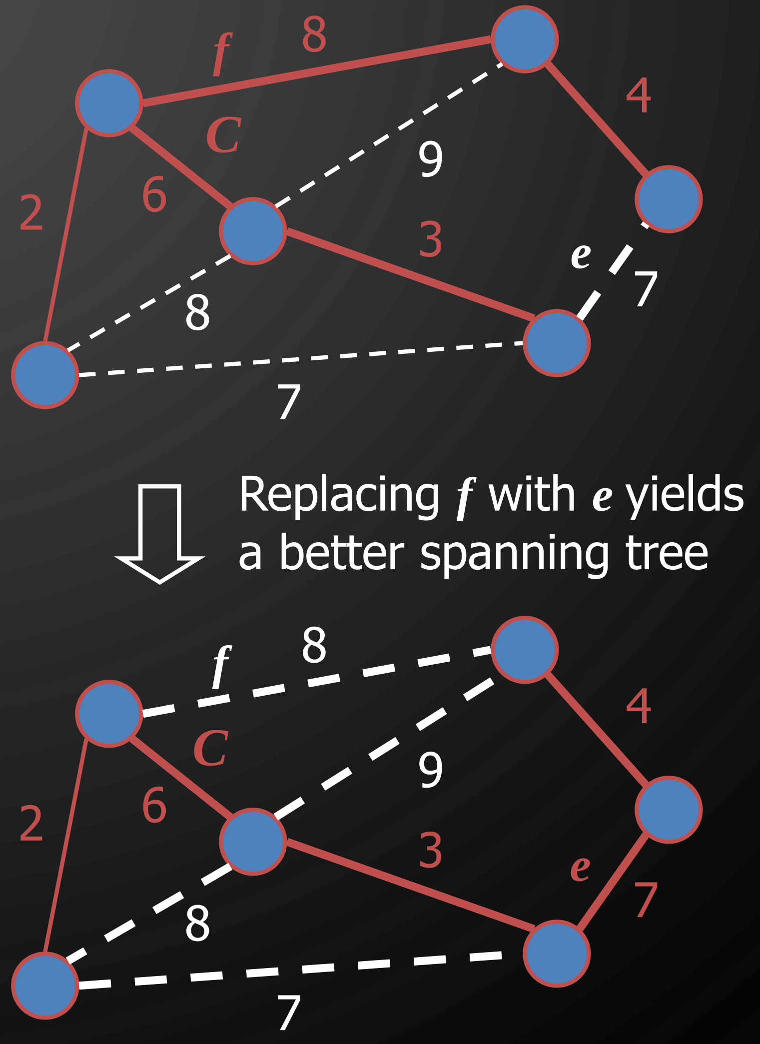
EXERCISE
MST

- Show an MST of the following graph.



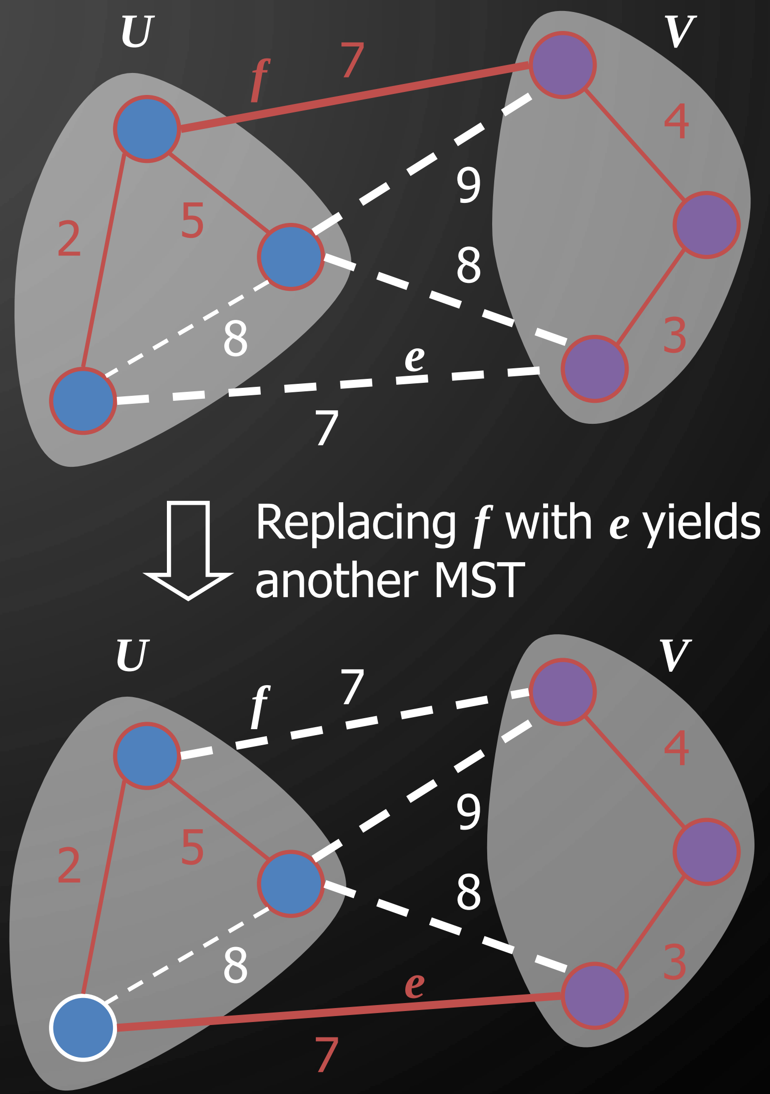
CYCLE PROPERTY

- **Cycle Property:**
  - Let T be a minimum spanning tree of a weighted graph G
  - Let e be an edge of G that is not in T and C let be the cycle formed by e with T
  - For every edge f of C,
    $weight(f) \leq weight(e)$
- Proof by contradiction:
  - If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f



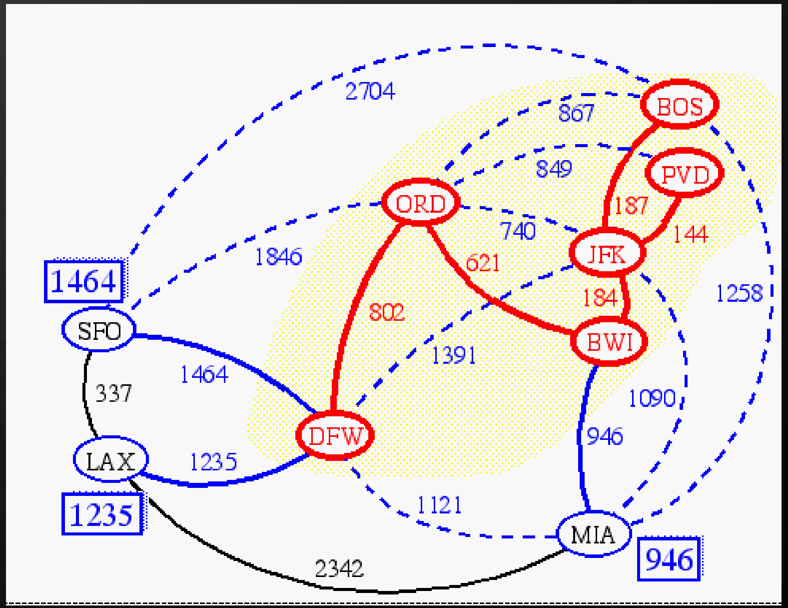
PARTITION PROPERTY

- **Partition Property:**
  - Consider a partition of the vertices of G into subsets U and V
  - Let e be an edge of minimum weight across the partition
  - There is a minimum spanning tree of G containing edge e
- Proof by contradition:
  - Let T be an MST of G
  - If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
  - By the cycle property,
    $$weight(f) \leq weight(e)$$
  - Thus, $weight(f) = weight(e)$
  - We obtain another MST by replacing f with e



PRIM-JARNIK'S ALGORITHM

- We pick an arbitrary vertex s and we grow the MST as a cloud of vertices, starting from s
- We store with each vertex v a label $d(v)$ representing the smallest weight of an edge connecting v to a vertex in the cloud
- At each step:
  - We add to the cloud the vertex u outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u

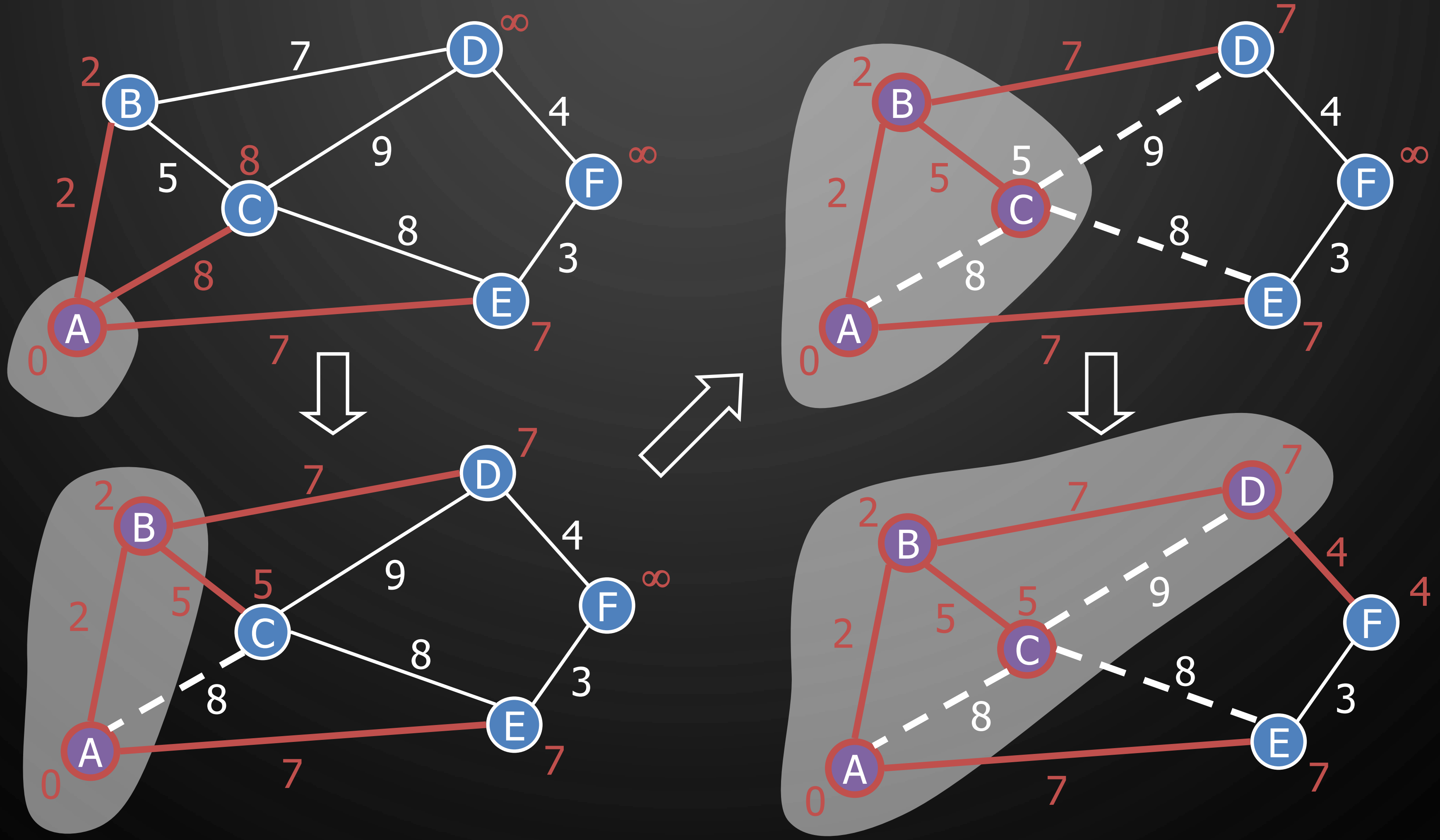


PRIM-JARNIK'S ALGORITHM

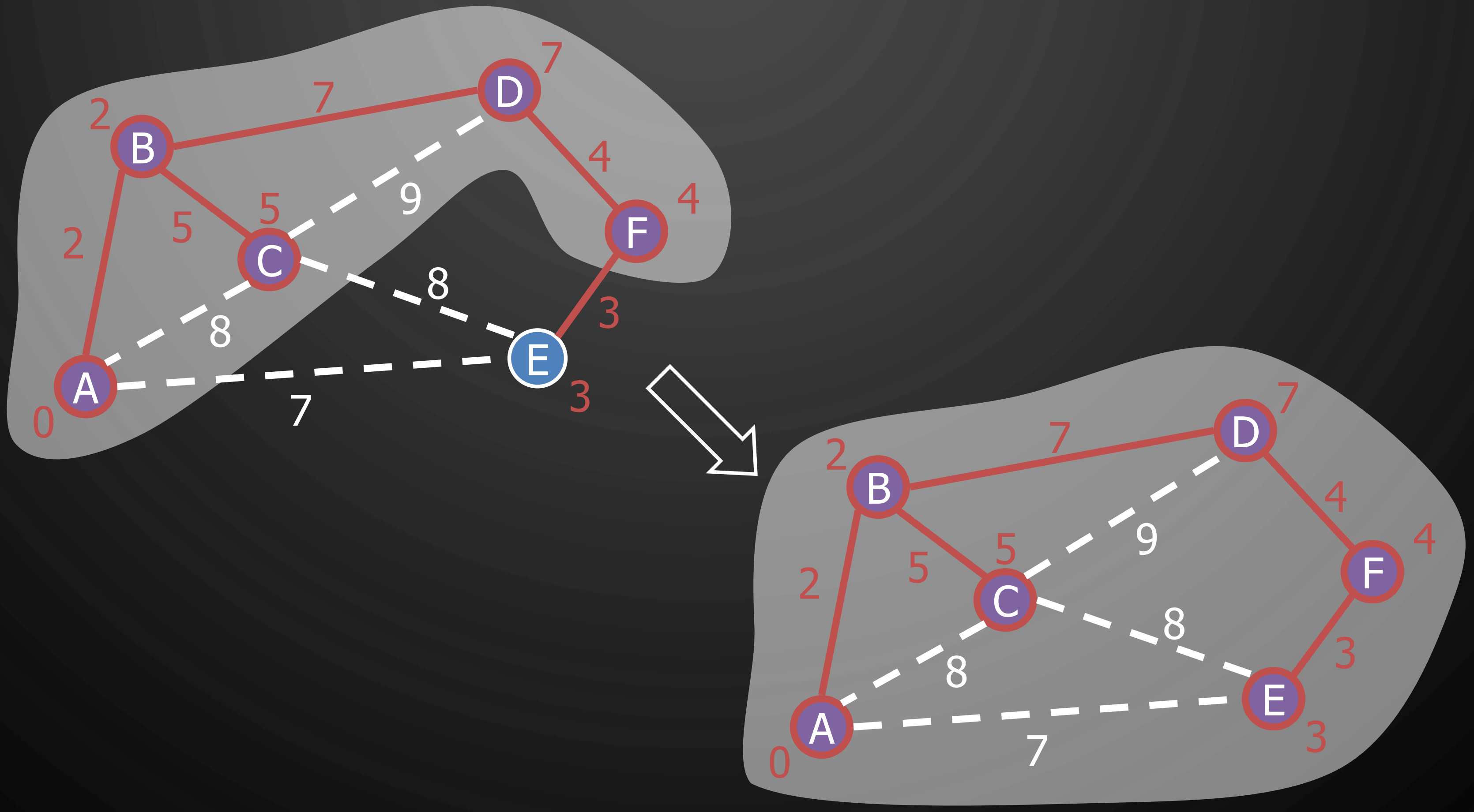
- An adaptable priority queue stores the vertices outside the cloud
  - Key: distance, $D[v]$
  - Element: vertex v
  - $Q.replace(i, k)$ changes the key of an item
- We store three labels with each vertex v:
  - Distance $D[v]$
  - Parent edge in MST $P[v]$
  - Locator in priority queue

```
Algorithm PrimJarnikMST(G)
Input: A weighted connected graph G
Output: A minimum spanning tree T of G
1.  Pick any vertex s of G
2.  D[s] ← 0;  P[s] ← ∅
3.  for each vertex v ≠ s do
4.     D[v] ← ∞;  P[v] ← ∅
5.  T ← ∅
6.  Priority queue Q of vertices with
    D[v] as the key
7.  while ¬Q.isEmpty() do
8.     u ← Q.removeMin()
9.     Add vertex u and edge P[u] to T
10.    for each e ∈ u.outgoingEdges() do
11.       v ← G.opposite(u, e)
12.       if e.weight() < D[v] then
13.          D[v] ← e.weight();  P[v] ← e
14.          Q.replace(v, D[v])
15. return T
```

EXAMPLE

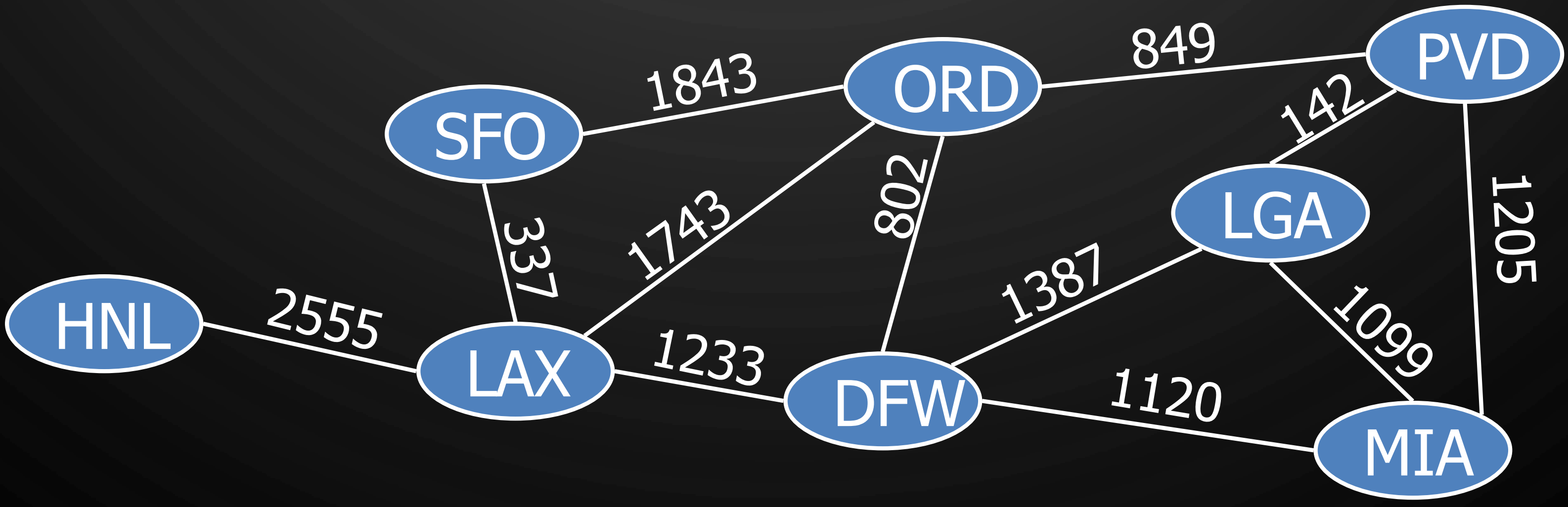

A B C D E F
0 2 8 ∞ 7 ∞
2 7 5 8 9 4 8 3 7
A B C D E F
0 2 5 7 7 ∞
2 7 5 8 9 4 8 3 7
A B C D E F
0 2 5 7 7 ∞
2 7 5 8 9 4 8 3 7
A B C D E F
0 2 5 7 7 4
2 7 5 8 9 4 8 3 7

EXAMPLE



EXERCISE
PRIM'S MST ALGORITHM

- Show how Prim's MST algorithm works on the following graph, assuming you start with SFO
  - Show how the MST evolves in each iteration.



ANALYSIS

- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the distance, parent and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Prim-Jarnik's algorithm runs in $O\big((n+m)\log n\big)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$
- If the graph is connected the running time is $O(m \log n)$