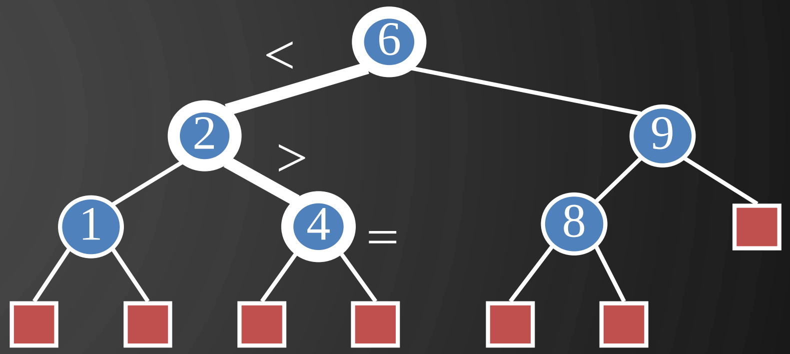


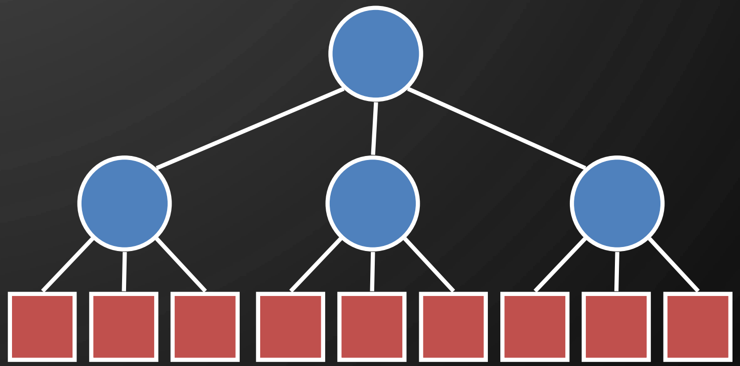
CHAPTER 12
SORTING AND SELECTION

ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN JAVA, GOODRICH, TAMASSIA AND GOLDWASSER (WILEY 2016)

DIVIDE AND CONQUER ALGORITHMS

DIVIDE AND CONQUER ALGORITHMS
ANALYSIS WITH RECURRENCE EQUATIONS

- Divide-and-conquer is a general algorithm design paradigm:
  - Divide: divide the input data S into k (disjoint) subsets $S_1, S_2, \ldots, S_k$
  - Recur: solve the subproblems recursively
  - Conquer: combine the solutions for $S_1, S_2, \ldots, S_k$ into a solution for S
- The base case for the recursion are subproblems of constant size
- Analysis can be done using recurrence equations (relations)

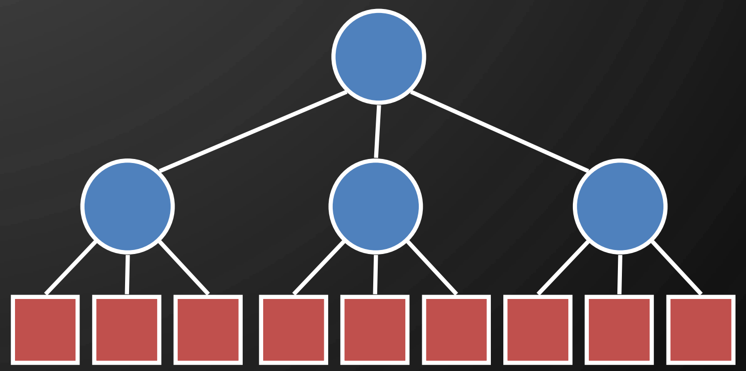


DIVIDE AND CONQUER ALGORITHMS
ANALYSIS WITH RECURRENCE EQUATIONS

- When the size of all subproblems is the same (frequently the case) the recurrence equation representing the algorithm is:

$$T(n) = D(n) + kT\left(\frac{n}{c}\right) + C(n)$$

- Where
  - $D(n)$ is the cost of dividing S into the k subproblems $S_1, S_2, \ldots, S_k$
  - There are k subproblems, each of size $\frac{n}{c}$ that will be solved recursively
  - $C(n)$ is the cost of combining the subproblem solutions to get the solution for S



EXERCISE
RECURRENCE EQUATION SETUP

- Algorithm – transform multiplication of two n-bit integers I and J into multiplication of $\left(\frac{n}{2}\right)$-bit integers and some additions/shifts

1. Where does recursion happen in this algorithm?
2. Rewrite the step(s) of the algorithm to show this clearly.

Algorithm multiply(I, J)

Input: n-bit integers I, J

Output: $I * J$

1. **if** $n > 1$ **then**
2. Split I and J into high and low order halves: I_h, I_l, J_h, J_l
3. $x_1 \leftarrow I_h * J_h$; $x_2 \leftarrow I_h * J_l$;
4. $x_3 \leftarrow I_l * J_h$; $x_4 \leftarrow I_l * J_l$
5. $Z \leftarrow x_1 * 2^n + x_2 * 2^{\frac{n}{2}} + x_3 * 2^{\frac{n}{2}} + x_4$
6. **else**
7. $Z \leftarrow I * J$
8. **return** Z

EXERCISE
RECURRENCE EQUATION SETUP

- Algorithm – transform multiplication of two n-bit integers I and J into multiplication of $\left(\frac{n}{2}\right)$-bit integers and some additions/shifts

3. Assuming that additions and shifts of n-bit numbers can be done in $O(n)$ time, describe a recurrence equation showing the running time of this multiplication algorithm

Algorithm multiply(I, J)
Input: n-bit integers I, J
Output: $I * J$
1. **if** $n > 1$ **then**
2. Split I and J into high and low order halves: I_h, I_l, J_h, J_l
3. $x_1 \leftarrow$multiply(I_h, J_h); $x_2 \leftarrow$multiply(I_h, J_l)
4. $x_3 \leftarrow$multiply(I_l, J_h); $x_4 \leftarrow$multiply(I_l, J_l)
5. $Z \leftarrow x_1 * 2^n + x_2 * 2^{\frac{n}{2}} + x_3 * 2^{\frac{n}{2}} + x_4$
6. **else**
7. $Z \leftarrow I * J$
8. **return** Z

EXERCISE
RECURRENCE EQUATION SETUP

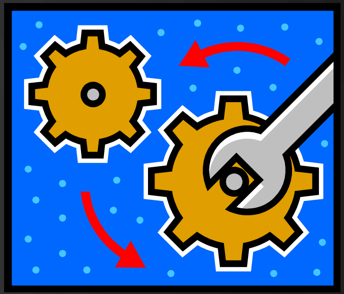
- Algorithm – transform multiplication of two n-bit integers I and J into multiplication of $\left(\frac{n}{2}\right)$-bit integers and some additions/shifts
- The recurrence equation for this algorithm is:
  - $T(n) = 4T\left(\frac{n}{2}\right) + O(n)$
  - The solution is $O(n^2)$ which is the same as naïve algorithm

```
Algorithm multiply(I, J)
Input: n-bit integers I, J
Output: I * J
1. if n > 1 then
2.     Split I and J into high
         and low order halves:
         I_h, I_l, J_h, J_l
3.     x_1 ← multiply(I_h, J_h);  x_2 ← multiply(I_h, J_l)
4.     x_3 ← multiply(I_l, J_h);  x_4 ← multiply(I_l, J_l)
5.     Z ← x_1 * 2^n + x_2 * 2^(n/2) + x_3 * 2^(n/2) + x_4
6. else
7.     Z ← I * J
8. return Z
```

DIVIDE AND CONQUER ALGORITHMS
ANALYSIS WITH RECURRENCE EQUATIONS

- Remaining question: how do we solve recurrence relations?
  - **Iterative substitution** – continually expand a recurrence to yield a summation, then bound the summation
  - **Analyze the recursion tree** – determine work per level and number of levels in a recursion tree. This is not a proof technique, more of an intuitive sketch of a proof
  - **Master theorem (method)** – rule to go directly to solution of recurrence. This is slightly beyond scope of course, but we will see it anyway

ITERATIVE SUBSTITUTION



- In the iterative substitution, or "plug-and-chug," technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern. Example:
  - $T(n) = 2T\left(\frac{n}{2}\right) + bn$
  - $\qquad = 2\left(2T\left(\frac{n}{2^2}\right) + b\left(\frac{n}{2}\right)\right) + bn = 2^2T\left(\frac{n}{2^2}\right) + 2bn$
  - $\qquad = 2^3T\left(\frac{n}{2^3}\right) + 3bn$
  - $\qquad = \cdots$
  - $\qquad = 2^iT\left(\frac{n}{2^i}\right) + ibn$
- Note that base, $T(n) = b$, case occurs when $2^i = n$. That is, $i = \log n$.
- So,

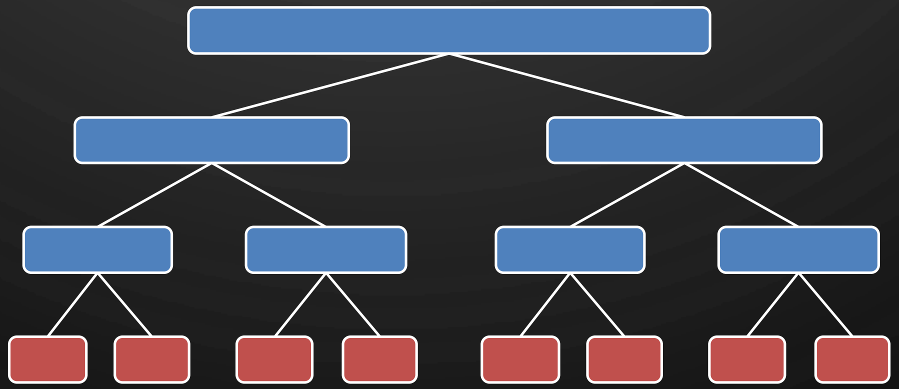
$$T(n) = bn + n\log n = O(n\log n)$$

THE RECURSION TREE

- Draw the recursion tree for the recurrence relation and look for a pattern.

  Example: $T(n) = 2T\left(\frac{n}{2}\right) + bn$

depth	T's	size	time
0	1	n	bn
1	2	$n/2$	bn
i	2^i	$n/2^i$	bn
…	…	…	…



- Total time: $bn + bn \log n = O(n \log n)$

THE MASTER THEOREM (METHOD)

- Many divide-and-conquer algorithms have the form:

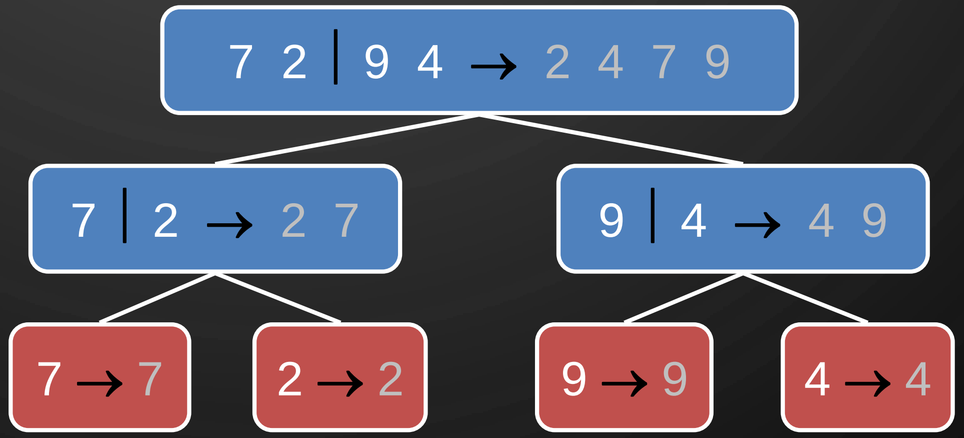
$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- The master theorem:
  1. If $f(n)$ is $O\left(n^{\log_b a-\epsilon}\right)$, then $T(n)$ is $\Theta\left(n^{\log_b a}\right)$
  2. If $f(n)$ is $\Theta\left(n^{\log_b a}\log^k n\right)$, then $T(n)$ is $\Theta\left(n^{\log_b a}\log^{k+1} n\right)$
  3. If $f(n)$ is $\Omega\left(n^{\log_b a+\epsilon}\right)$, then $T(n)$ is $\Theta\left(f(n)\right)$,

     provided $af\left(\frac{n}{b}\right) \leq \delta f(n)$ for some $\delta < 1$

- Examples
- $T(n) = 4T\left(\frac{n}{2}\right) + n$
  - $O(n^2)$
- $T(n) = T\left(\frac{n}{2}\right) + 1$
  - $O(\log n)$, (binary search)
- $T(n) = T\left(\frac{n}{3}\right) + n\log n$
  - $O(n\log n)$

MERGE SORT



MERGE-SORT

- **Merge-sort** is based on the divide-and-conquer paradigm. It consists of three steps:
    - Divide: partition input sequence S into two sequences S_1 and S_2 of about $\frac{n}{2}$ elements each
    - Recur: recursively sort S_1 and S_2
    - Conquer: merge S_1 and S_2 into a sorted sequence
- What is the recurrence relation?

Algorithm mergeSort(S, C)
Input: Sequence S of n elements,
Comparator C
Output: Sequence S sorted according to C

1. **if** S.size() > 1 **then**
2. $(S_1, S_2) \leftarrow$ partition$\left(S, \frac{n}{2}\right)$
3. $S_1 \leftarrow$ mergeSort(S_1, C)
4. $S_2 \leftarrow$ mergeSort(S_2, C)
5. $S \leftarrow$ merge(S_1, S_2)
6. **return** S

MERGE-SORT

- The running time of Merge Sort can be expressed by the recurrence equation:

$$T(n) = 2T\left(\frac{n}{2}\right) + M(n)$$

- We need to determine $M(n)$, the time to merge two sorted sequences each of size $\frac{n}{2}$.

Algorithm mergeSort(S, C)
Input: Sequence S of n elements,
    Comparator C
Output: Sequence S sorted according to C

1. **if** S.size() > 1 **then**
2. $(S_1, S_2) \leftarrow$ partition$\left(S, \frac{n}{2}\right)$
3. $S_1 \leftarrow$ mergeSort(S_1, C)
4. $S_2 \leftarrow$ mergeSort(S_2, C)
5. $S \leftarrow$ merge(S_1, S_2)
6. **return** S

MERGING TWO SORTED SEQUENCES

- The conquer step of merge-sort consists of merging two sorted sequences A and B into a sorted sequence S containing the union of the elements of A and B
- Merging two sorted sequences, each with $\frac{n}{2}$ elements and implemented by means of a doubly linked list, takes $O(n)$ time
  - $M(n) = O(n)$

Algorithm merge(A, B)

Input: Sequences A, B with $\frac{n}{2}$ elements each

Output: Sorted sequence of $A \cup B$

```
S ← ∅
while ¬A.isEmpty() ∧ ¬B.isEmpty() do
    if A.first() < B.first() then
        S.addLast(A.removeFirst())
    else
        S.addLast(B.removeFirst())
while ¬A.isEmpty() do
    S.addLast(A.removeFirst())
while ¬B.isEmpty() do
    S.addLast(B.removeFirst())
return S
```

MERGESORT

- So, the running time of Merge Sort can be expressed by the recurrence equation:

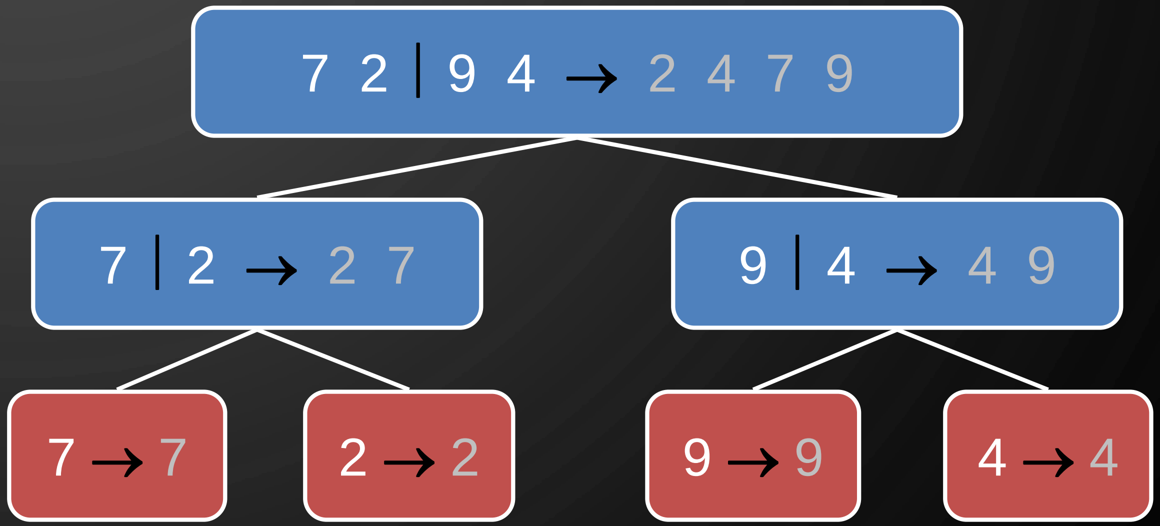
$$T(n) = 2T\left(\frac{n}{2}\right) + M(n)$$
$$= 2T\left(\frac{n}{2}\right) + O(n)$$
$$= O(n \log n)$$

Algorithm mergeSort(S, C)
Input: Sequence S of n elements,
Comparator C
Output: Sequence S sorted according to C

1. **if** S.size() > 1 **then**
2. $(S_1, S_2) \leftarrow$ partition$\left(S, \frac{n}{2}\right)$
3. $S_1 \leftarrow$ mergeSort(S_1, C)
4. $S_2 \leftarrow$ mergeSort(S_2, C)
5. $S \leftarrow$ merge(S_1, S_2)
6. **return** S

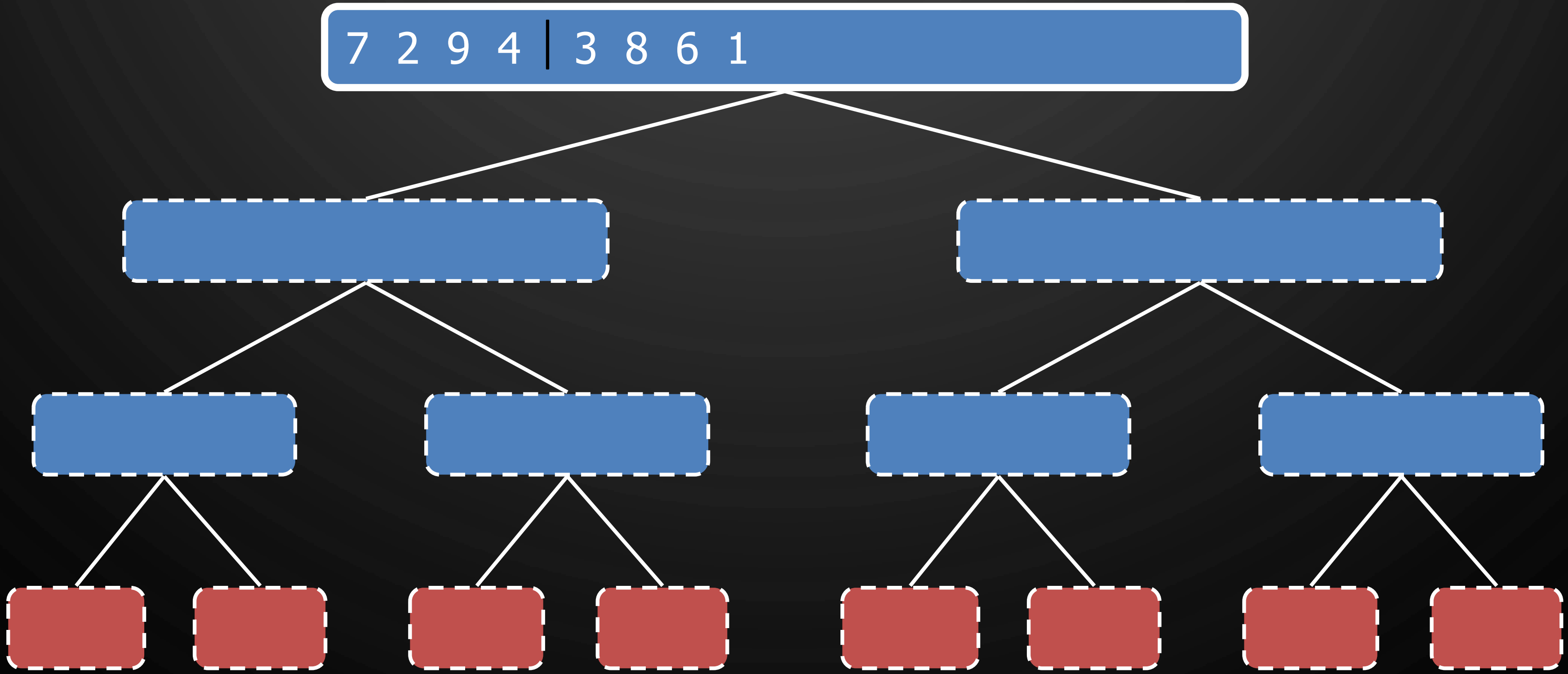
MERGE-SORT EXECUTION TREE (RECURSIVE CALLS)

- An execution of merge-sort is depicted by a binary tree
  - Each node represents a recursive call of merge-sort and stores
    - Unsorted sequence before the execution and its partition
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



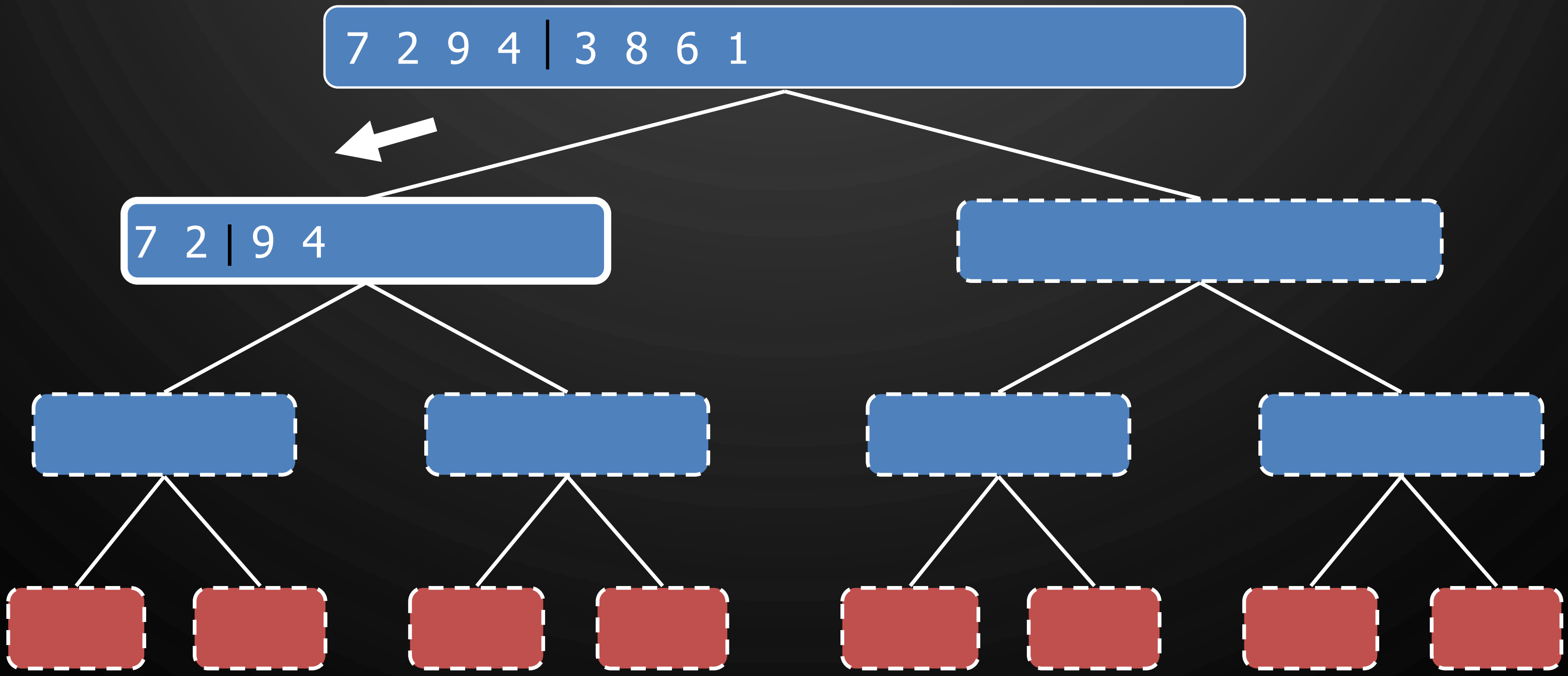
EXECUTION EXAMPLE

- Partition



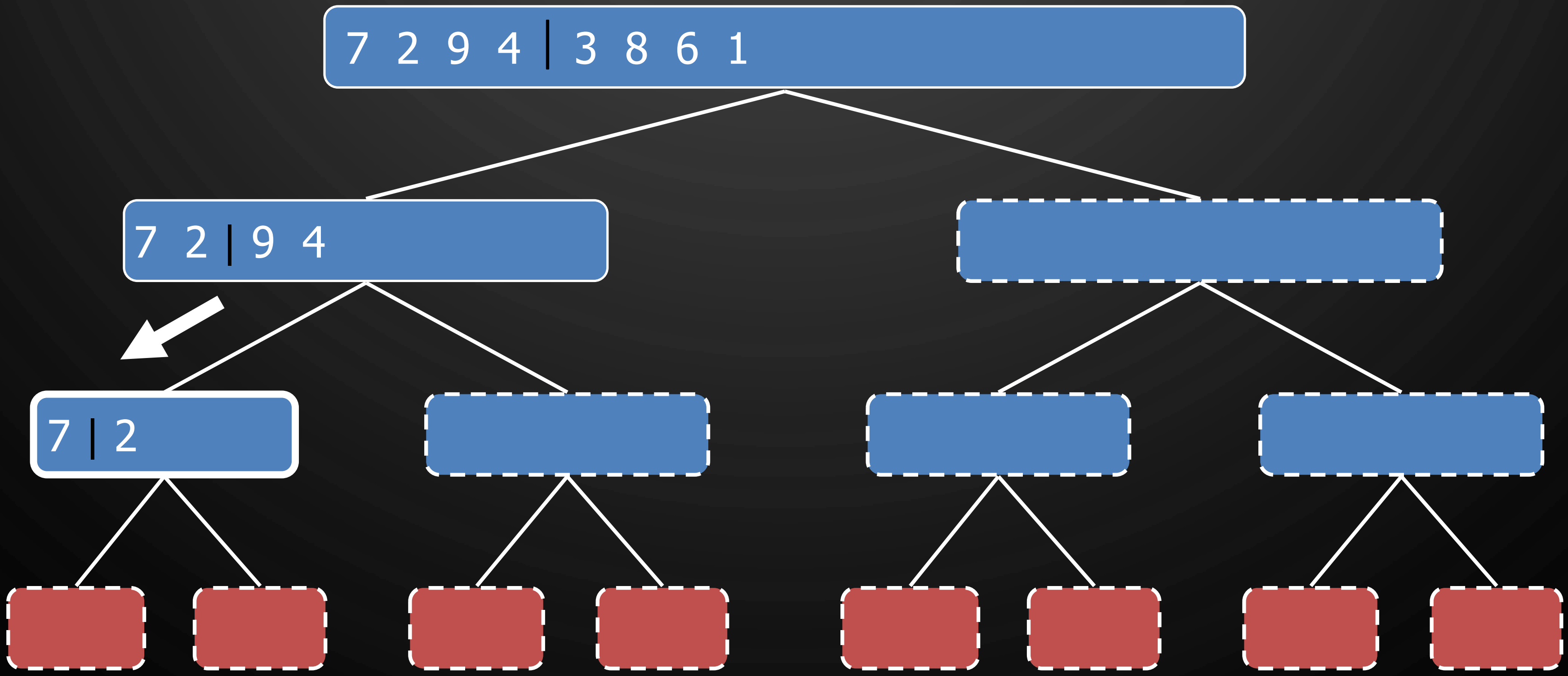
EXECUTION EXAMPLE

- Recursive Call, partition



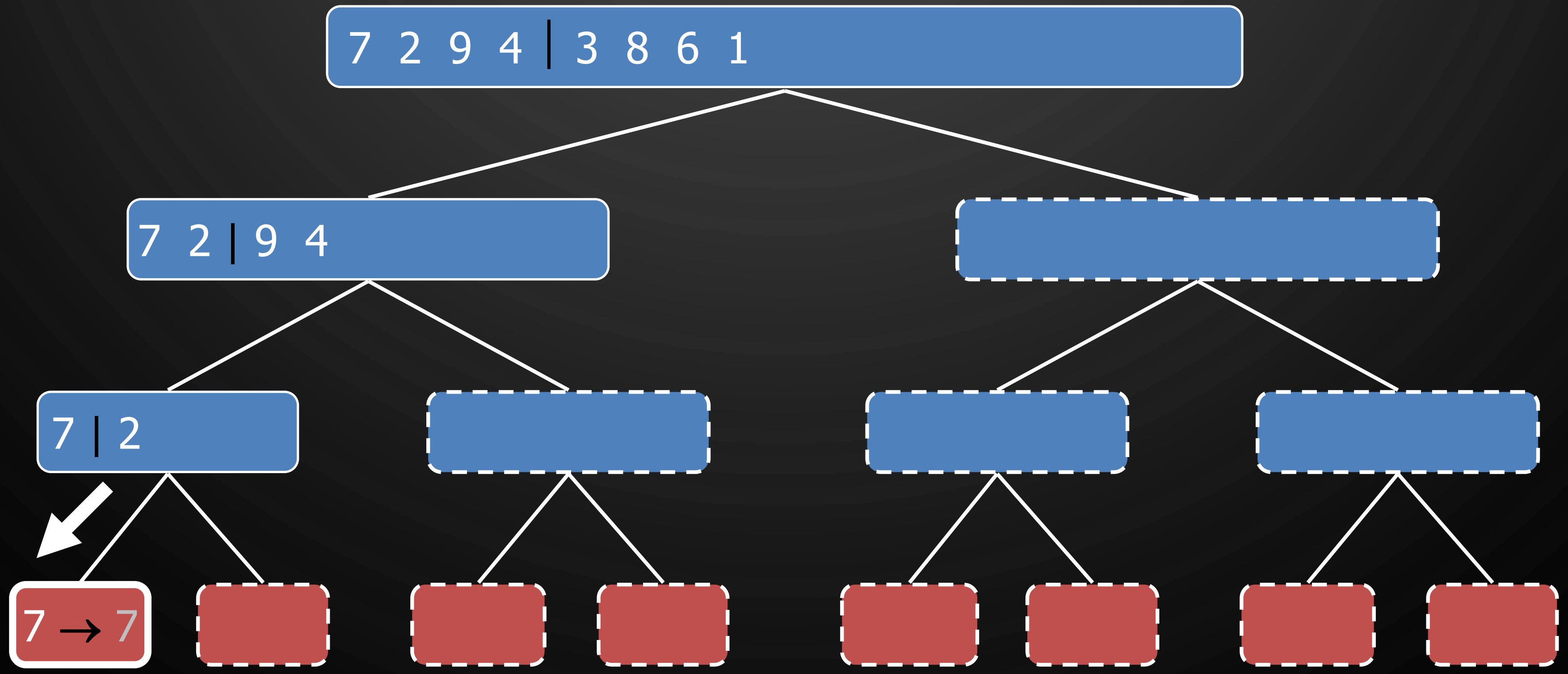
EXECUTION EXAMPLE

- Recursive Call, partition



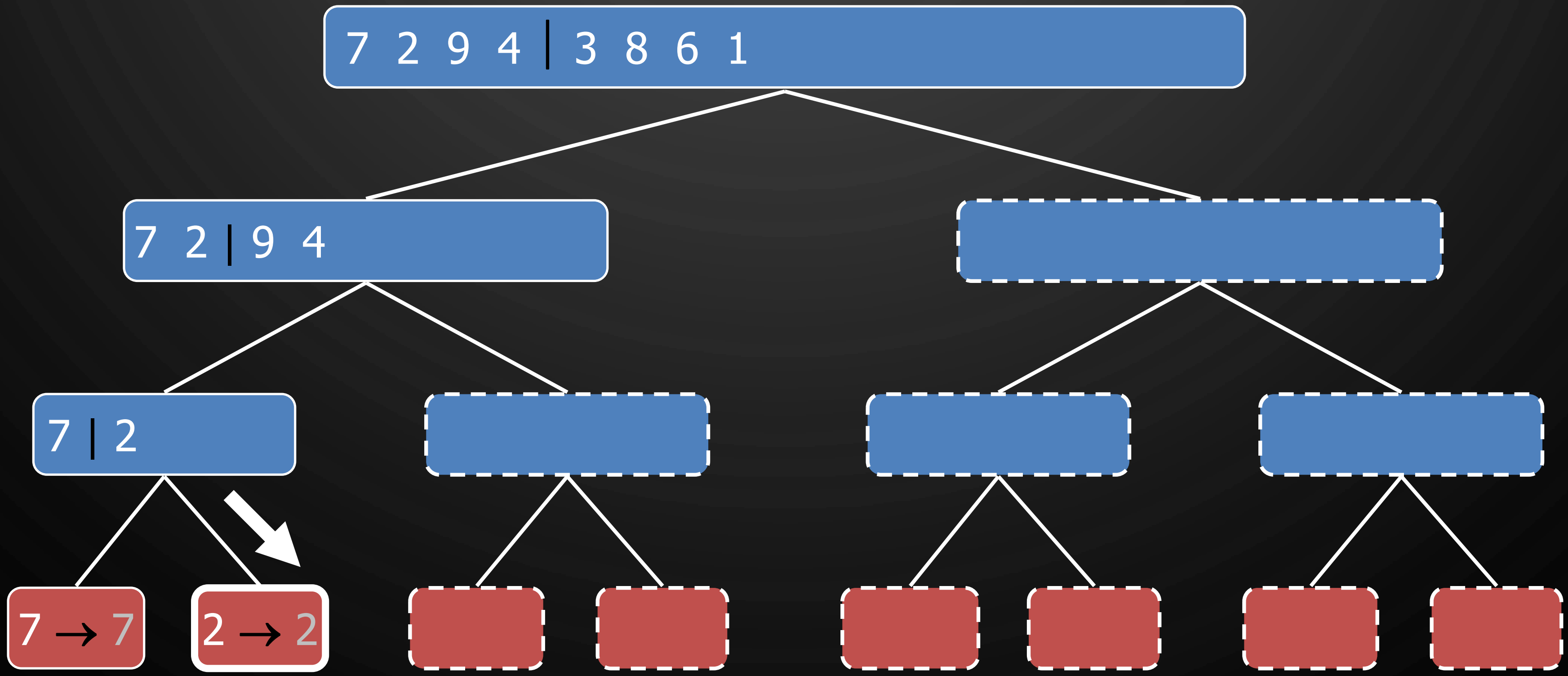
EXECUTION EXAMPLE

- Recursive Call, base case



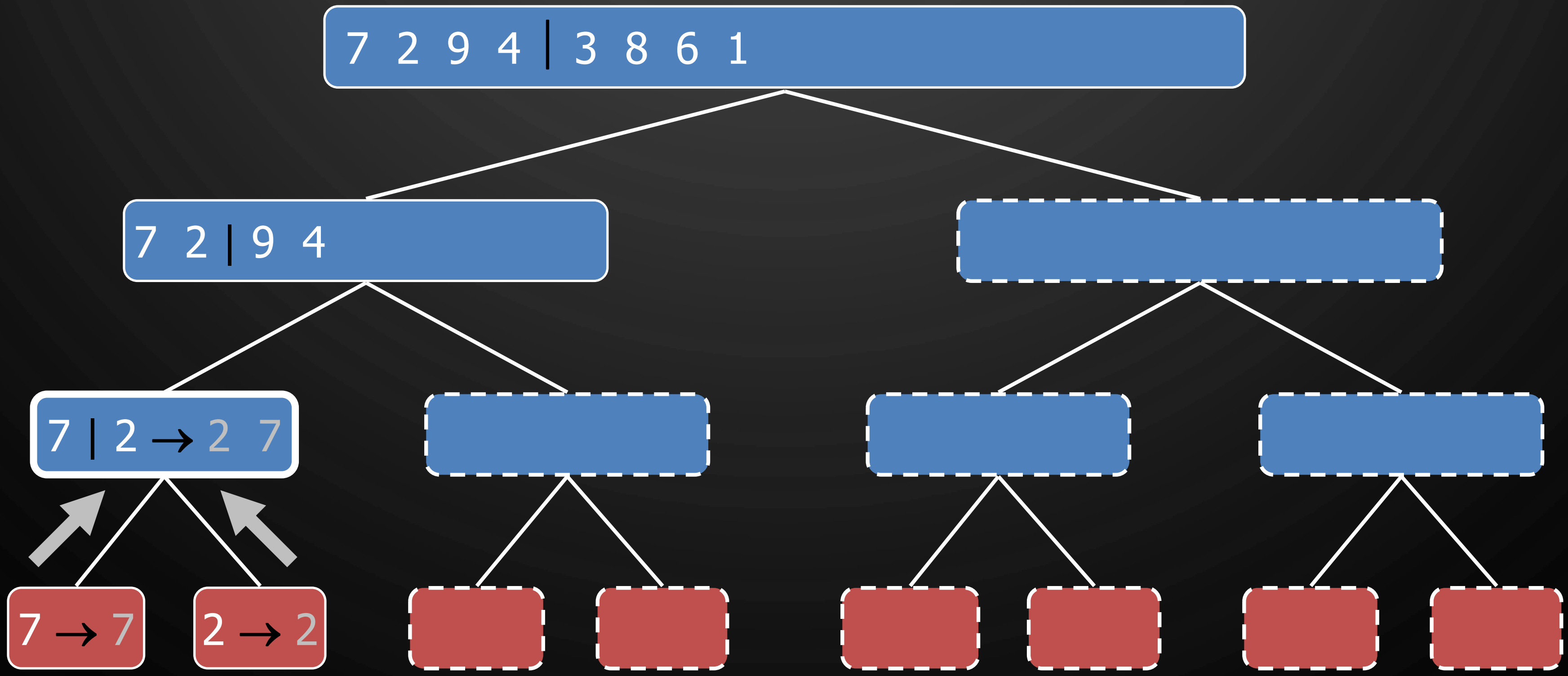
EXECUTION EXAMPLE

- Recursive Call, base case



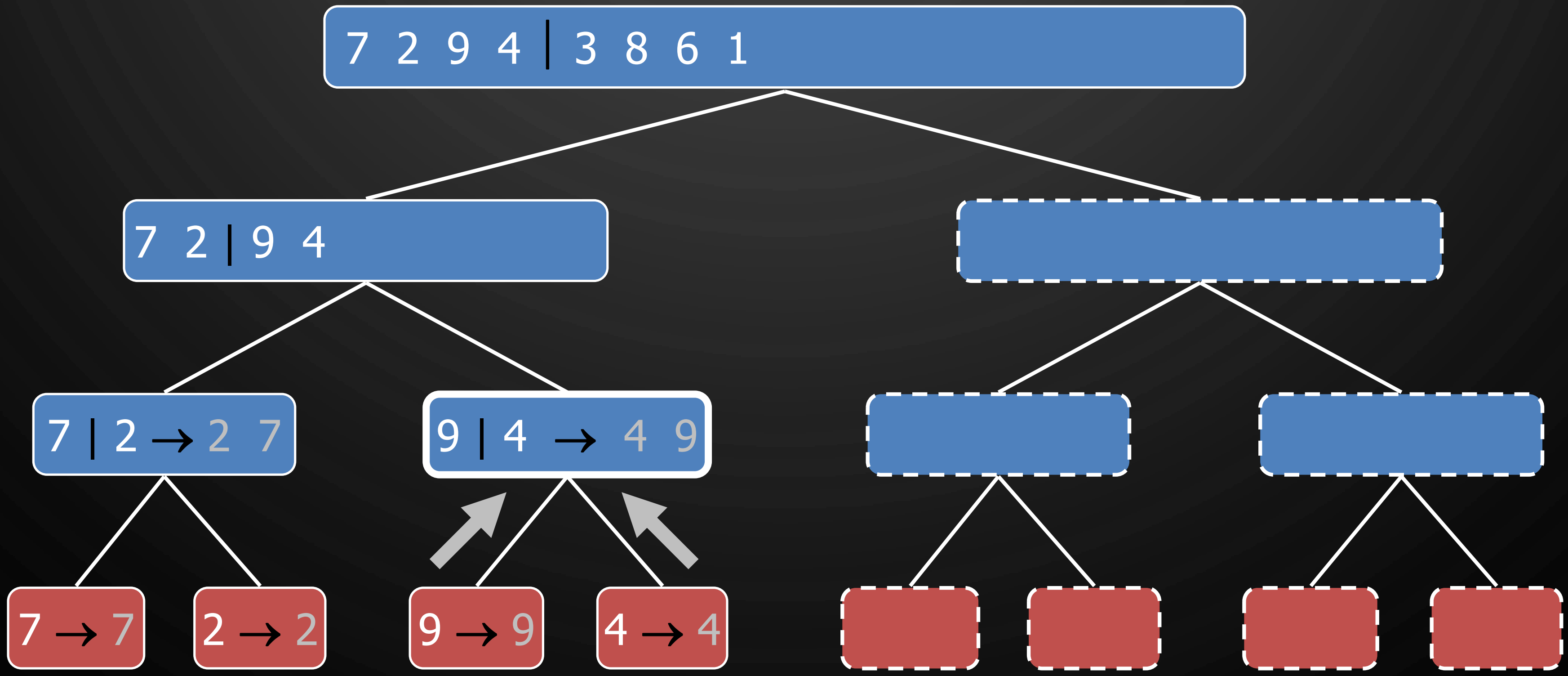
EXECUTION EXAMPLE

- Merge



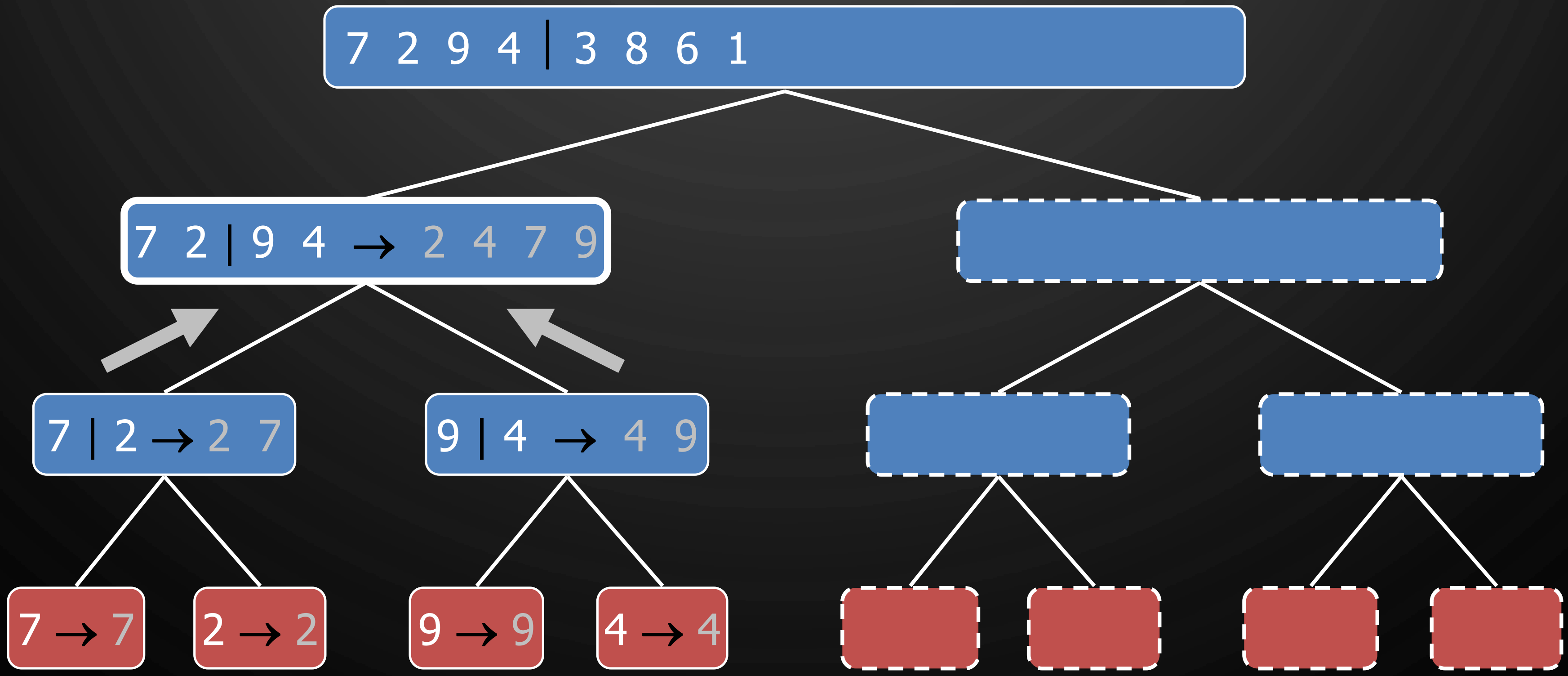
EXECUTION EXAMPLE

- Recursive call, …, base case, merge



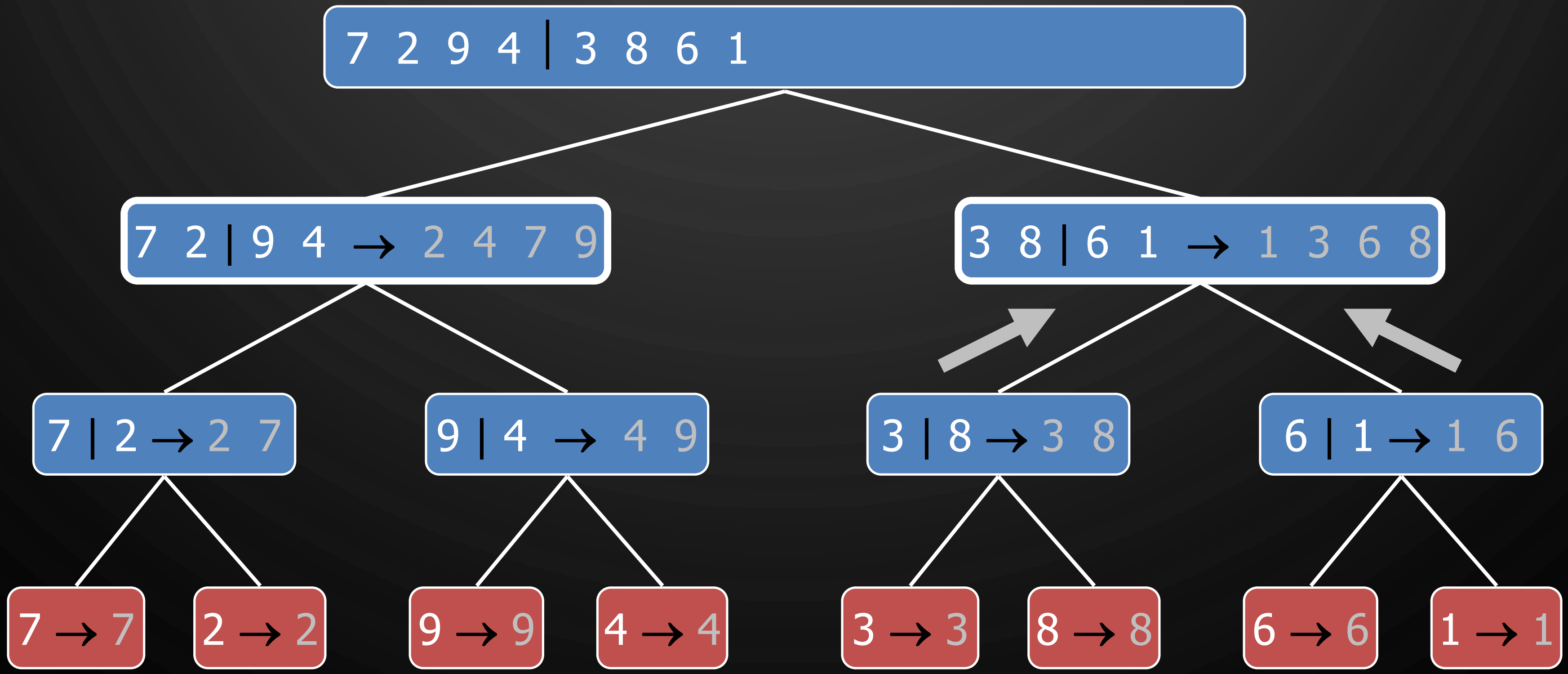
EXECUTION EXAMPLE

- Merge



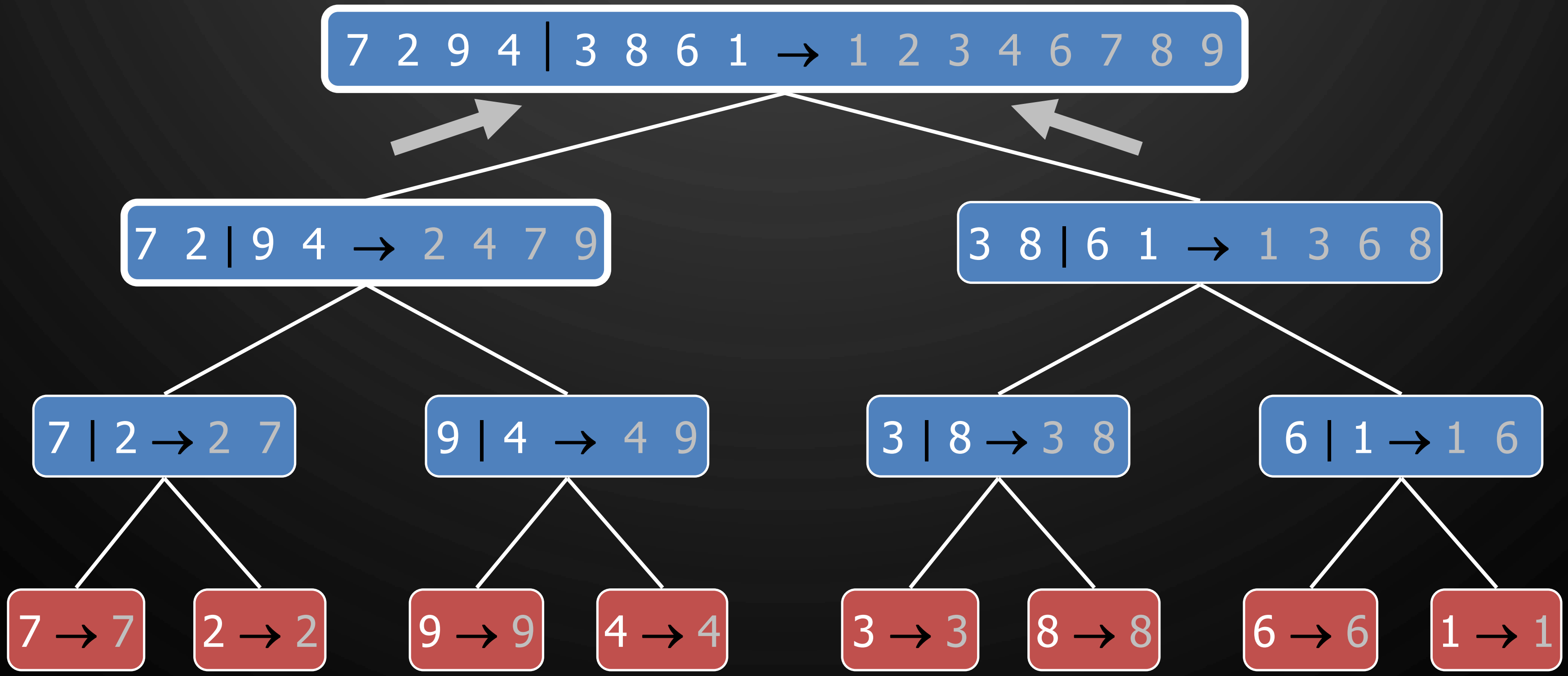
EXECUTION EXAMPLE

- Recursive call, …, merge, merge



EXECUTION EXAMPLE

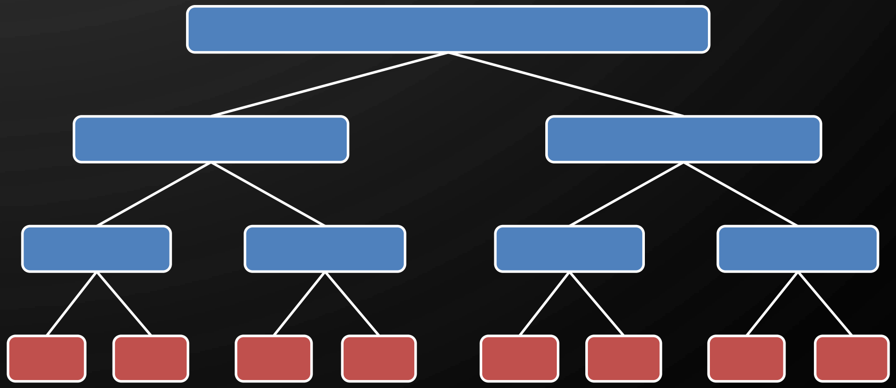
- Merge



ANOTHER ANALYSIS OF MERGE-SORT

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide in half the sequence,
- The work done at each level is $O(n)$
  - At level i, we partition and merge 2^i sequences of size $\frac{n}{2^i}$
- Thus, the total running time of merge-sort is $O(n \log n)$

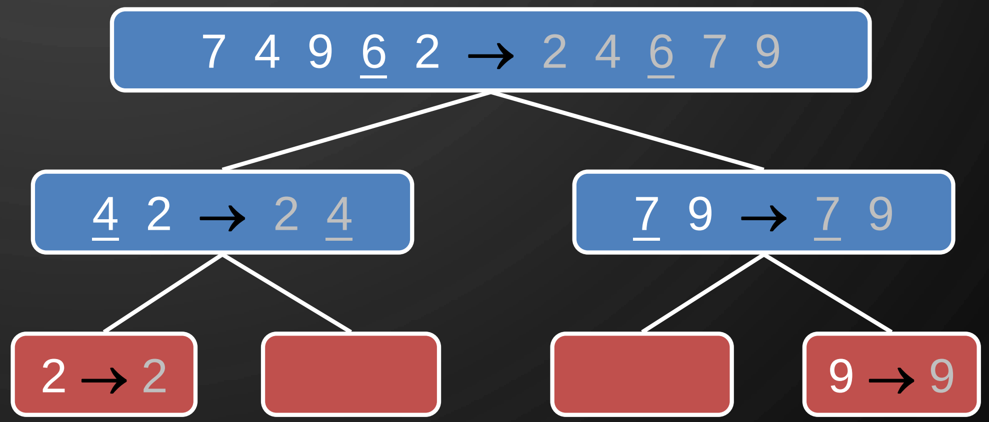
depth	#seqs	size	Cost for level
0	1	n	n
1	2	n/2	n
…	…	…	
i	2^i	$\frac{n}{2^i}$	n
…	…	…	
$\log n$	$2^{\log n} = n$	$\frac{n}{2^{\log n}} = 1$	n



SUMMARY OF SORTING ALGORITHMS (SO FAR)

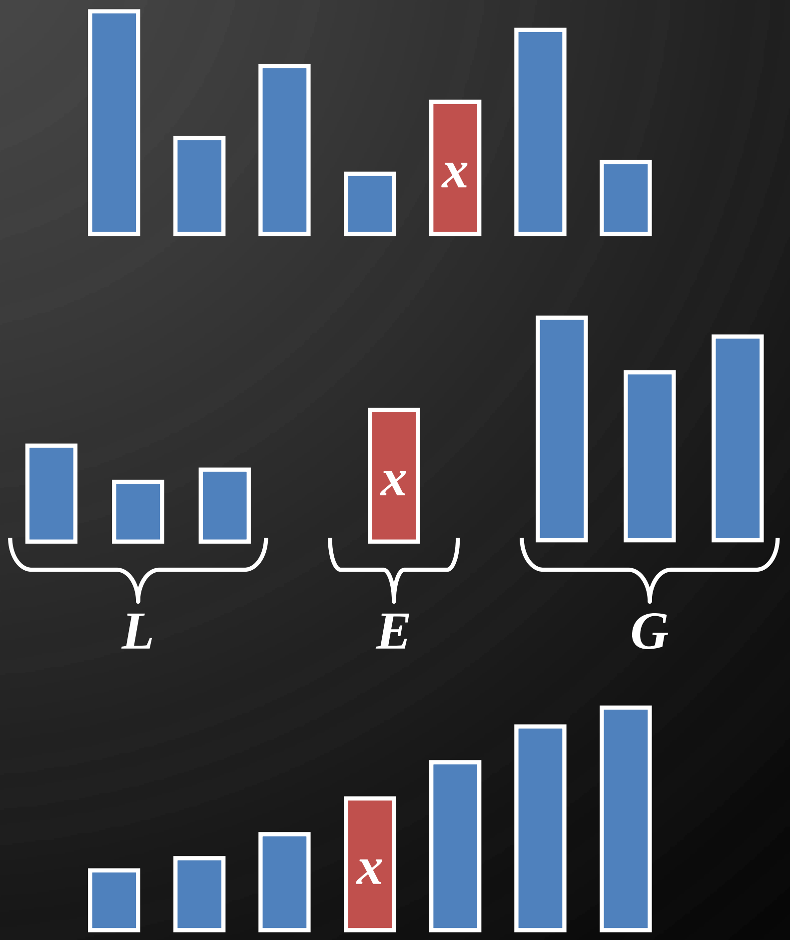
Algorithm	Time	Notes
Selection Sort	$O(n^2)$	Slow, in-place For small data sets (< 1K)
Insertion Sort	$O(n^2)$ WC, AC $O(n)$ BC	Slow, in-place For small data sets (< 1K)
Heap Sort	$O(n \log n)$	Fast, in-place For large data sets (1K – 1M)
Merge Sort	$O(n \log n)$	Fast, sequential data access For huge data sets (>1M)

QUICK-SORT



QUICK-SORT

- **Quick-sort** is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called **pivot**) and partition S into
    - L - elements less than x
    - E - elements equal x
    - G - elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E, and G



ANALYSIS OF QUICK SORT USING RECURRENCE RELATIONS

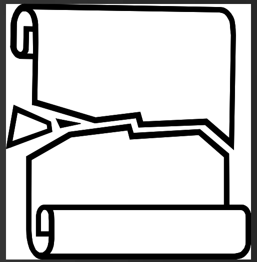
- Assumption: random pivot expected to give equal sized sublists
- The running time of Quick Sort can be expressed as:

$$T(n) = 2T\left(\frac{n}{2}\right) + P(n)$$

- $P(n)$ - time to partition on input of size n

```
Algorithm quickSort(S)
Input: Sequence S
Output: Sequence S with the elements
        sorted
if S.size() ≤ 1 then
  return S
i ← rand()%(r − l) + l //random integer
                       //between l and r
x ← S.at(i)
(L, E, G) ← partition(x)
quickSort(L)
quickSort(G)
return splice(L, E, G)
```

PARTITION



- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E, or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

```
Algorithm partition(S,p)
Input: Sequence S, position p of the pivot
Output: Subsequences L,E,G of the elements of S
            less than, equal to, or greater
            than the pivot, respectively
L,E,G ← ∅
x ← S.remove(p)
while ¬S.isEmpty() do
  y ← S.removeFirst()
  if y < x then
    L.addLast(y)
  else if y = x then
    E.addLast(y)
  else //y > x
    G.addLast(y)
return L,E,G
```

SO, THE EXPECTED COMPLEXITY OF QUICK SORT

- Assumption: random pivot expected to give equal sized sublists
- The running time of Quick Sort can be expressed as:

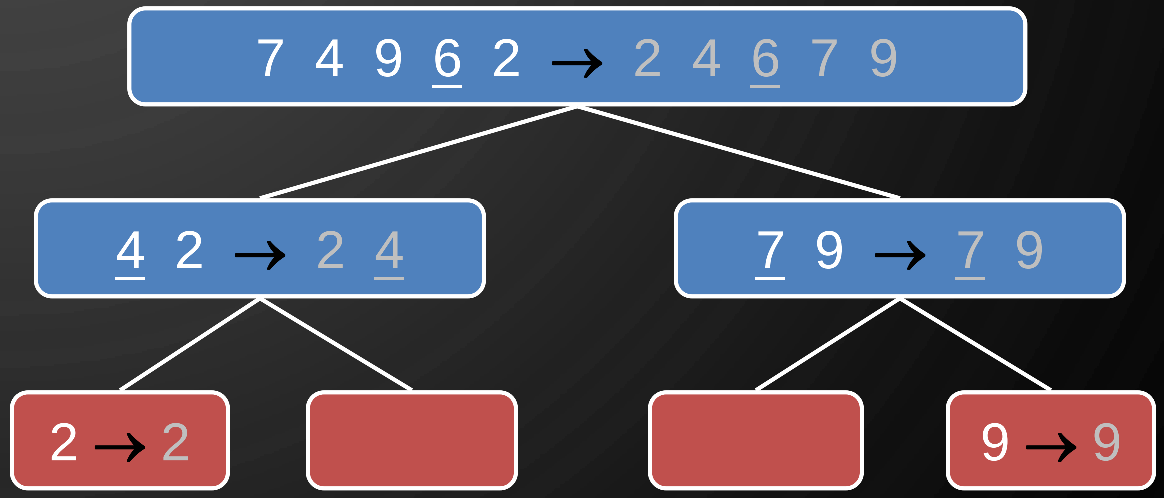
$$
\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + P(n) \\
&= 2T\left(\frac{n}{2}\right) + O(n) \\
&= O(n \log n)
\end{aligned}
$$

Algorithm quickSort(S)
Input: Sequence S
Output: Sequence S with the elements sorted

```
if S.size() ≤ 1 then
    return S
i ←rand()%(r − l) + l   //random integer
                        //between l and r
x ← S.at(i)
(L, E, G) ←partition(x)
quickSort(L)
quickSort(G)
return splice(L, E, G)
```

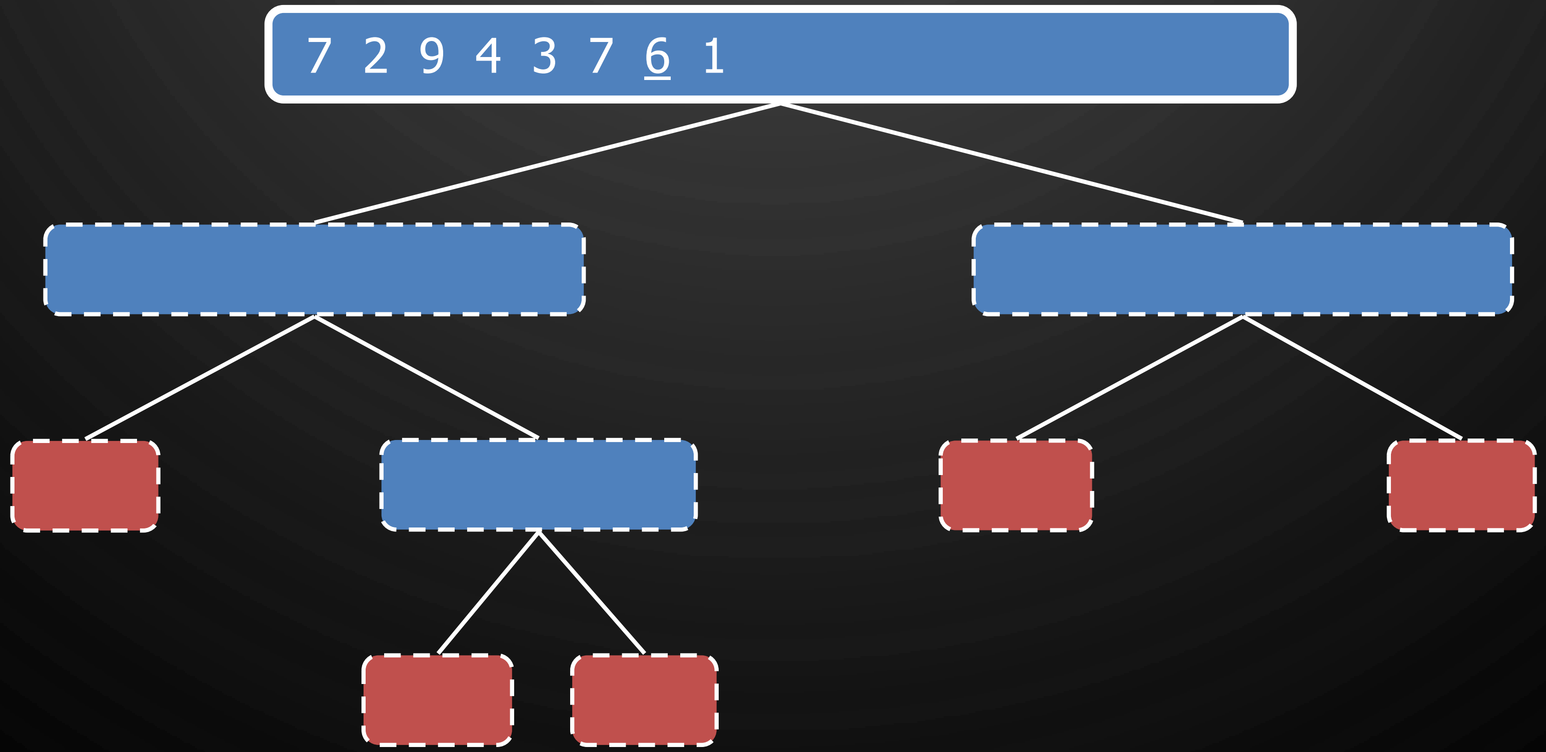
QUICK-SORT TREE

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



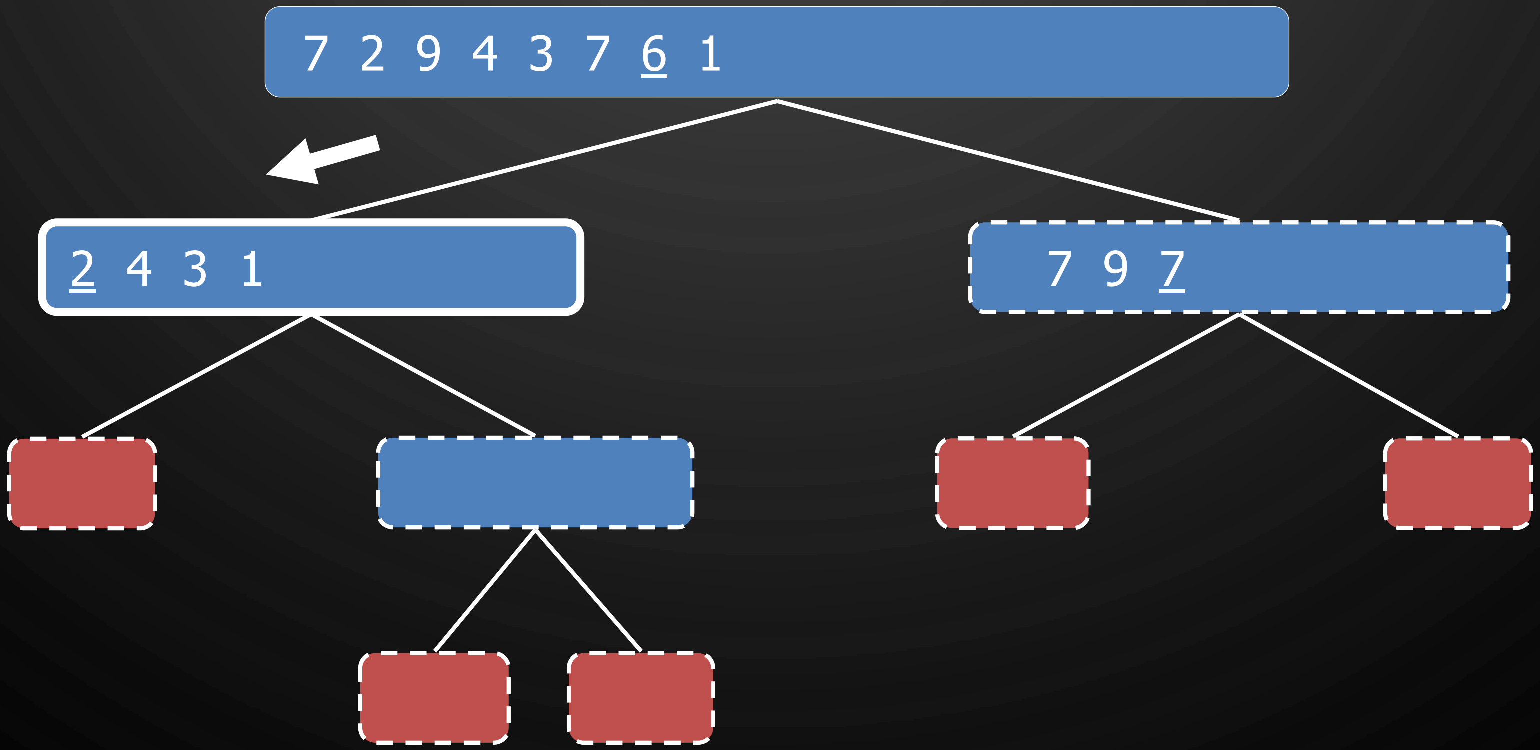
EXECUTION EXAMPLE

- Pivot selection



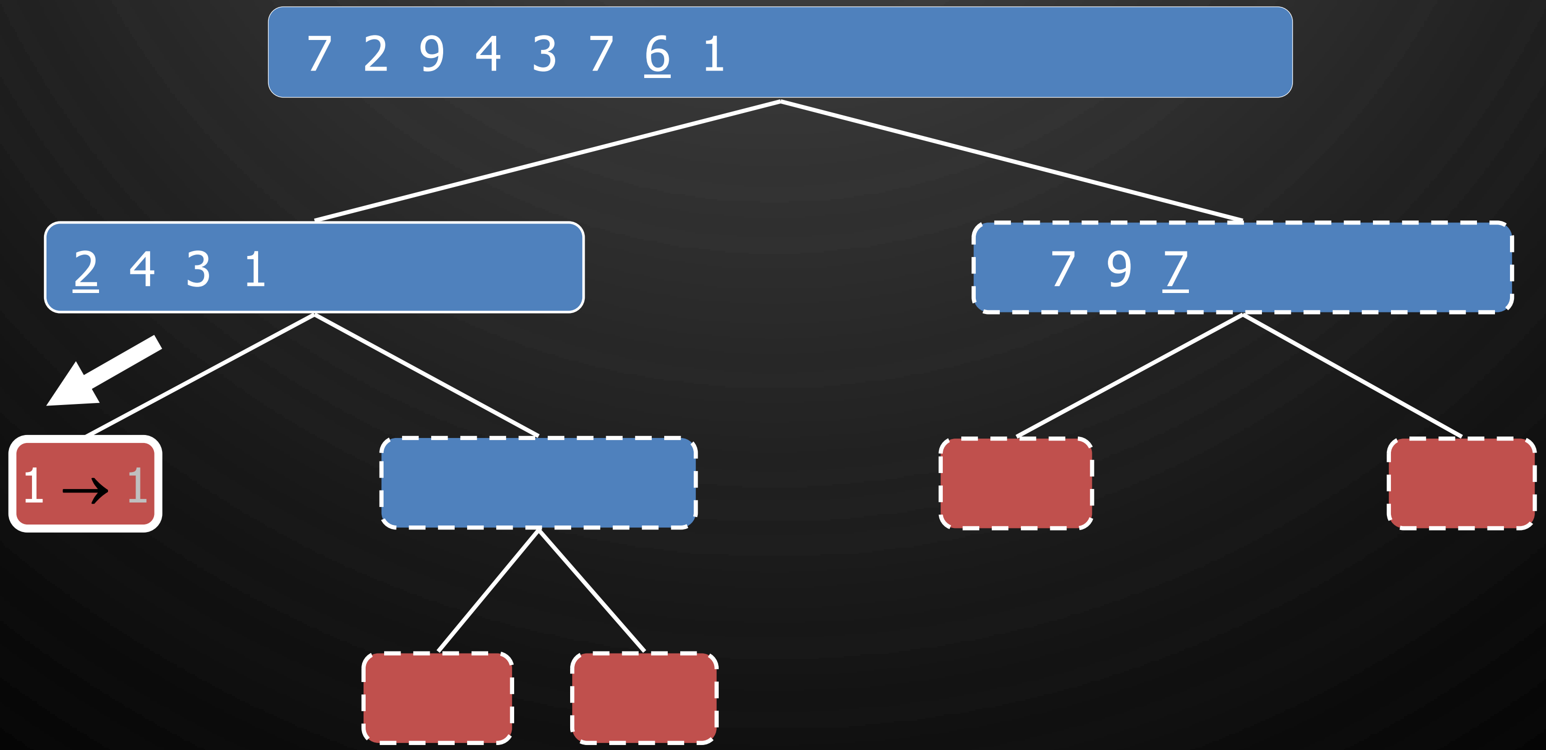
EXECUTION EXAMPLE

- Partition, recursive call, pivot selection



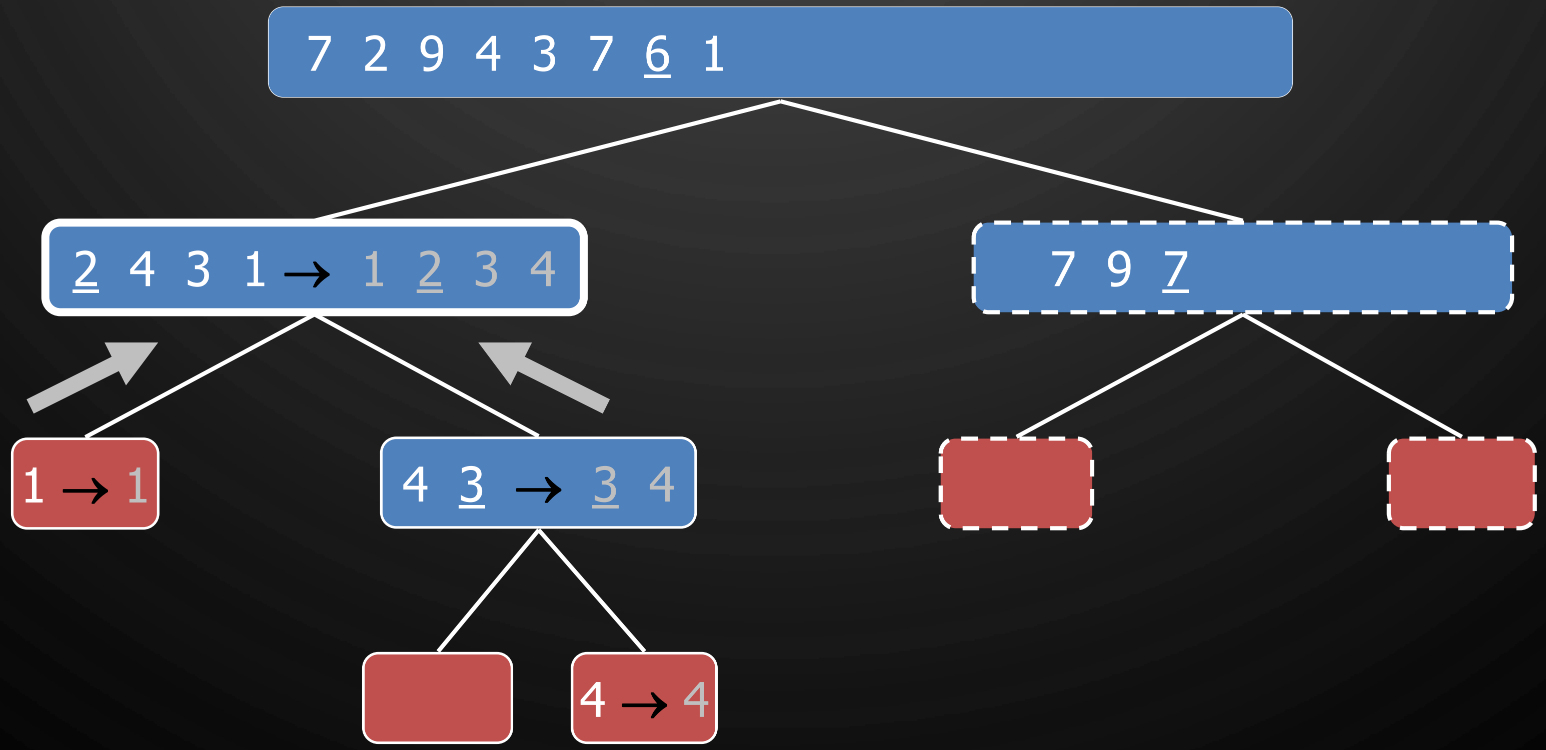
EXECUTION EXAMPLE

- Partition, recursive call, base case



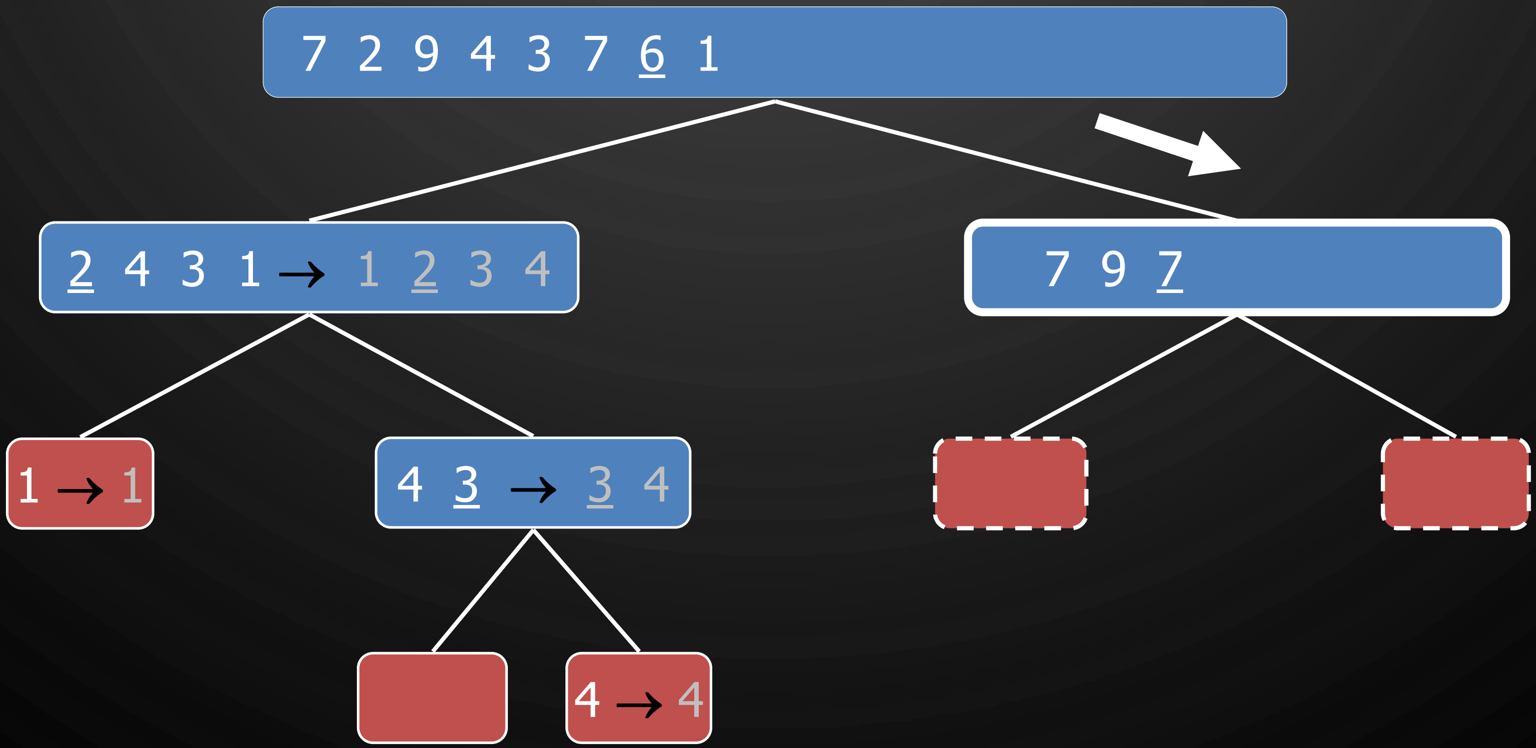
EXECUTION EXAMPLE

- Recursive call, …, base case, join



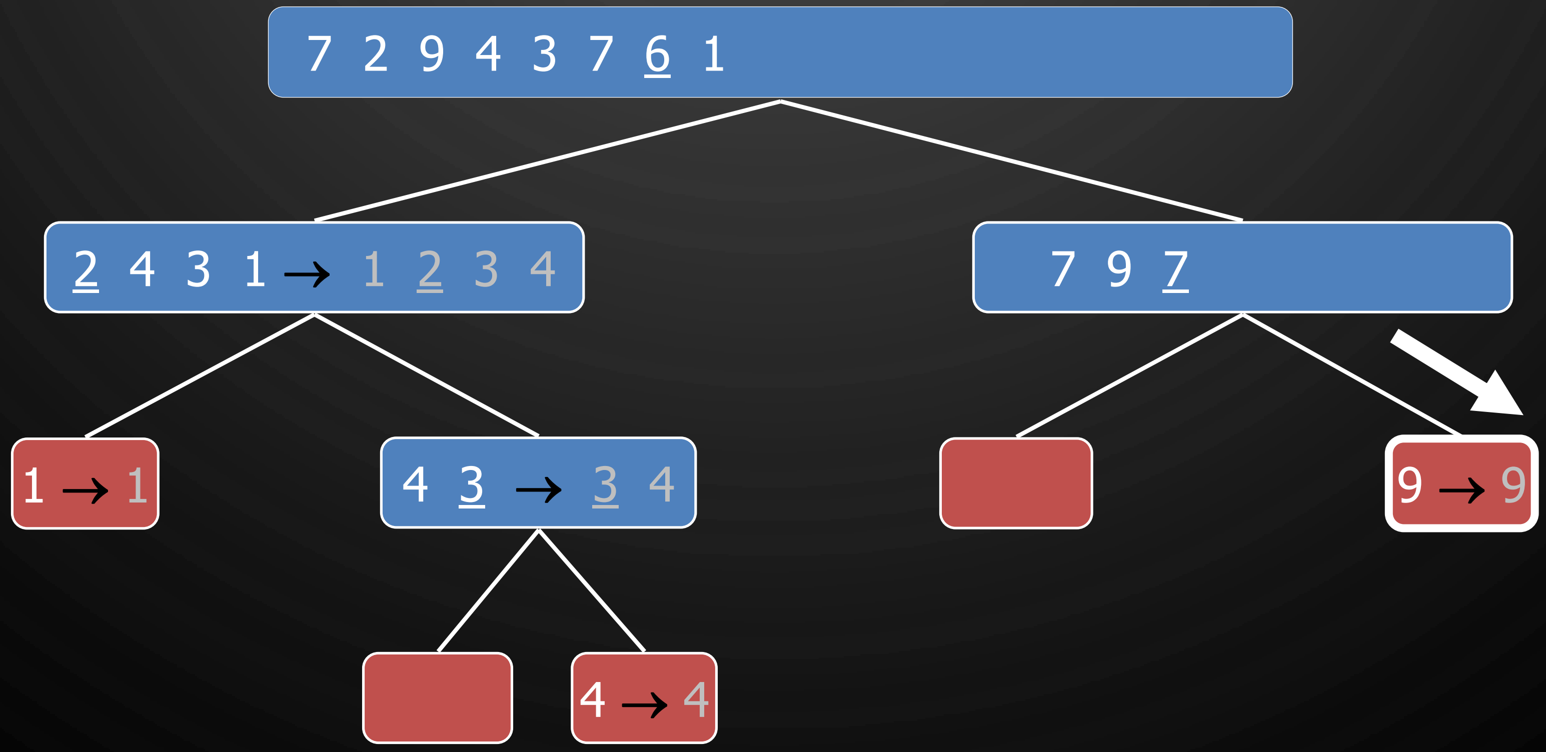
EXECUTION EXAMPLE

- Recursive call, pivot selection



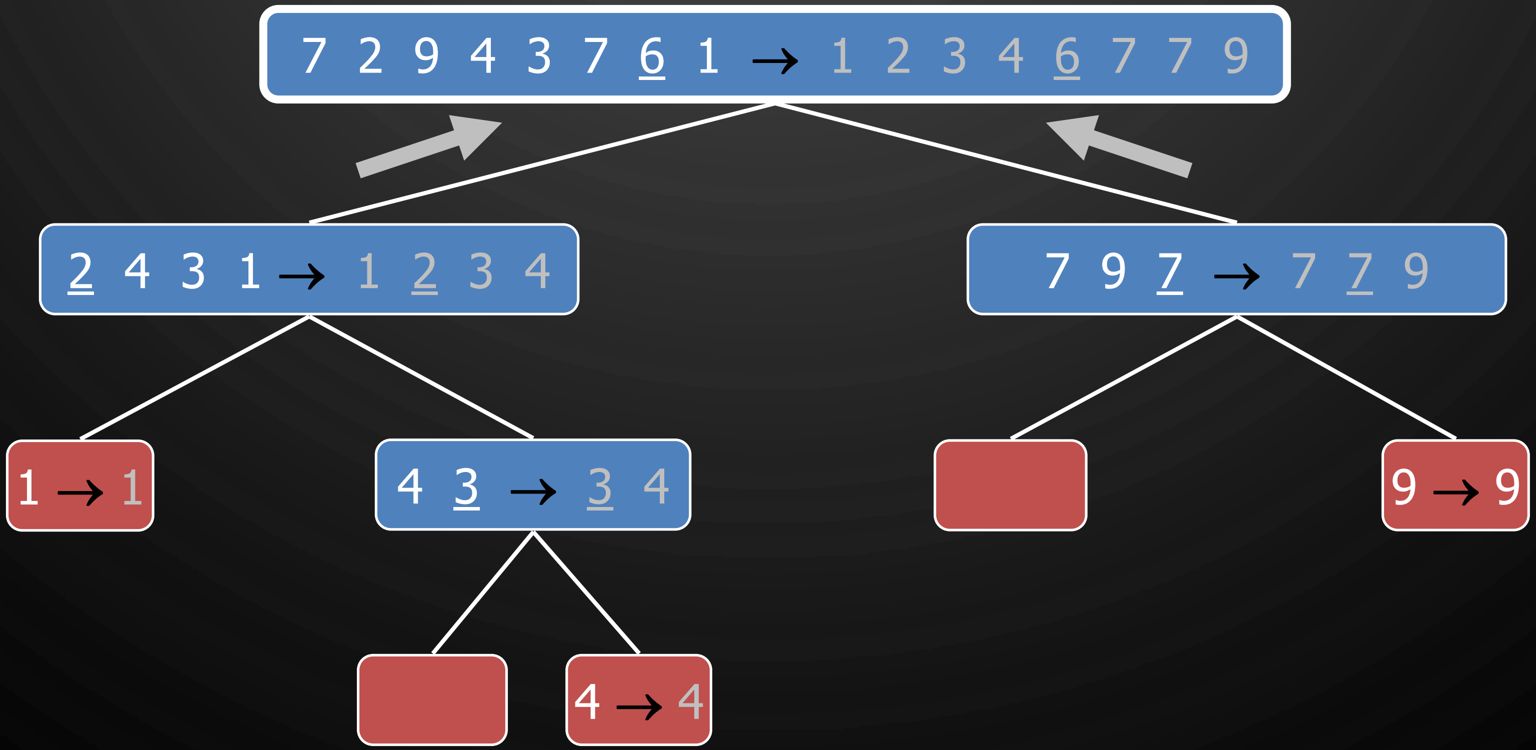
EXECUTION EXAMPLE

- Partition, …, recursive call, base case



EXECUTION EXAMPLE

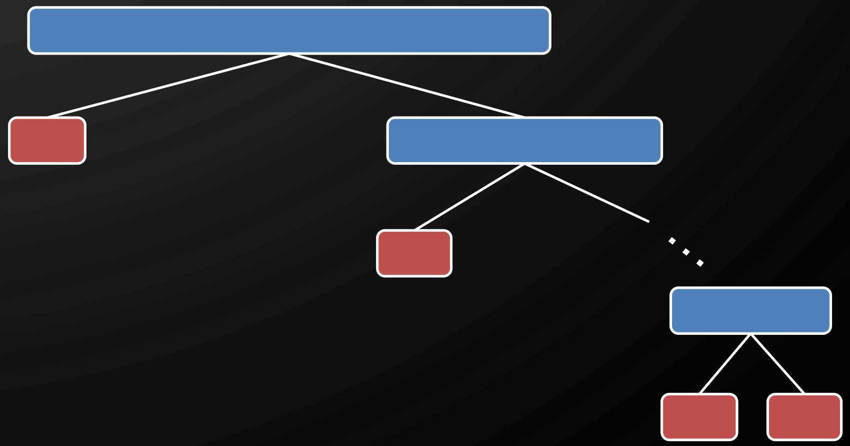
- Join, join



WORST-CASE RUNNING TIME

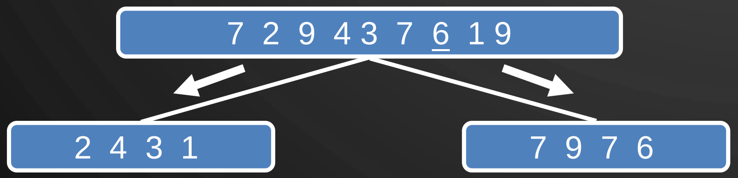
- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
  - One of L and G has size $n-1$ and the other has size 0
- The running time is proportional to:
  $$n + (n-1) + \cdots + 2 + 1 = O(n^2)$$
- Alternatively, using recurrence equations:
  $$T(n) = T(n-1) + O(n) = O(n^2)$$

depth	time
0	n
1	$n-1$
…	…
$n-1$	1

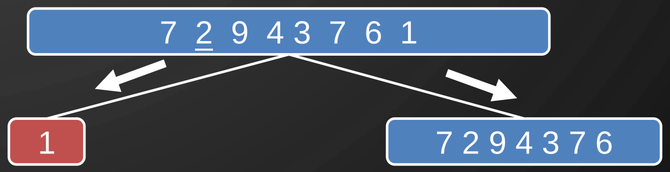


EXPECTED RUNNING TIME
REMOVING EQUAL SPLIT ASSUMPTION

- Consider a recursive call of quick-sort on a sequence of size s
  - Good call: the sizes of L and G are each less than $\frac{3s}{4}$
  - Bad call: one of L and G has size greater than $\frac{3s}{4}$

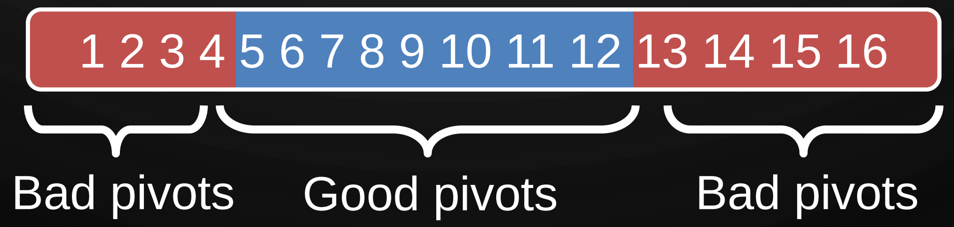


Good call



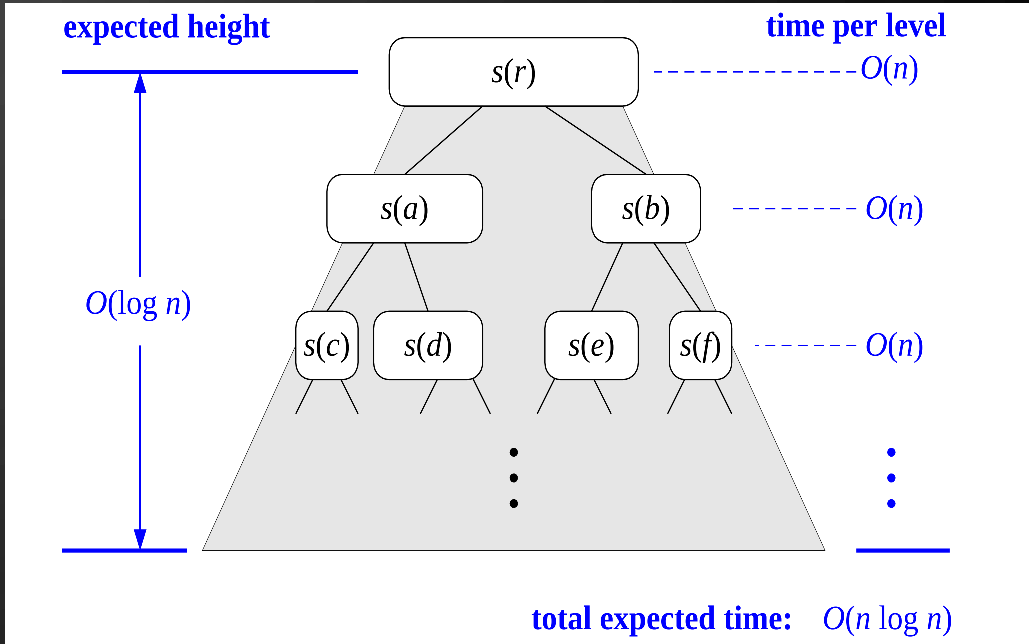
Bad call

- A call is good with probability $1/2$
  - $1/2$ of the possible pivots cause good calls:



EXPECTED RUNNING TIME

- Probabilistic Fact: The expected number of coin tosses required in order to get k heads is $2k$ (e.g., it is expected to take 2 tosses to get heads)
- For a node of depth i, we expect
  - $\frac{i}{2}$ ancestors are good calls
  - The size of the input sequence for the current call is at most $\left(\frac{3}{4}\right)^{\frac{i}{2}} n$
- Therefore, we have
  - For a node of depth $2\log_{\frac{4}{3}} n$, the expected input size is one
  - The expected height of the quick-sort tree is $O(\log n)$
- The amount or work done at the nodes of the same depth is $O(n)$
- Thus, the expected running time of quick-sort is $O(n \log n)$

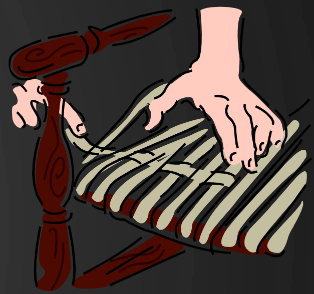


IN-PLACE QUICK-SORT

- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have indices less than h
  - the elements equal to the pivot have indices between h and k
  - the elements greater than the pivot have indices greater than k
- The recursive calls consider
  - elements with indices less than h
  - elements with indices greater than k

```
Algorithm inPlaceQuickSort(S, l, r)
Input: Array S, indices l,r
Output: Array S with the elements between
        l and r sorted
if l ≥ r then
   return S
i ←rand()%(r − l) + l  //random integer
                       //between l and r
x ← S[i]
(h, k) ←inPlacePartition(x)
inPlaceQuickSort(S, l, h − 1)
inPlaceQuickSort(S, k + 1, r)
return S
```

IN-PLACE PARTITIONING



- Perform the partition using two indices to split S into L and $E \cup G$ (a similar method can split $E \cup G$ into E and G).

j k

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9 (pivot = 6)

- Repeat until j and k cross:
  - Scan j to the right until finding an element $\geq x$.
  - Scan k to the left until finding an element $< x$.
  - Swap elements at indices j and k

j k

3 2 5 1 0 $\boxed{7}$ 3 5 9 $\boxed{2}$ 7 9 8 9 7 **6** 9

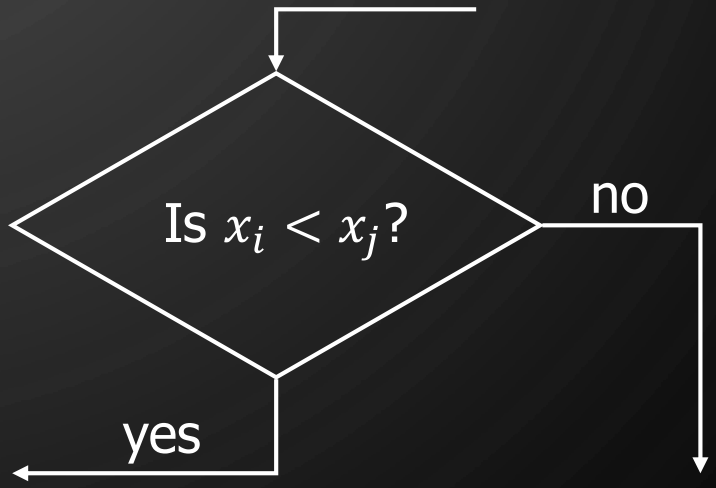
SUMMARY OF SORTING ALGORITHMS (SO FAR)

Algorithm	Time	Notes
Selection Sort	$O(n^2)$	In-place Slow, for small data sets
Insertion Sort	$O(n^2)$ WC, AC $O(n)$ BC	In-place Slow, for small data sets
Heap Sort	$O(n \log n)$	In-place Fast, For large data sets
Quick Sort	Exp. $O(n \log n)$ AC, BC $O(n^2)$ WC	Randomized, in-place Fastest, for large data sets
Merge Sort	$O(n \log n)$	Sequential data access Fast, for huge data sets

SORTING LOWER BOUND

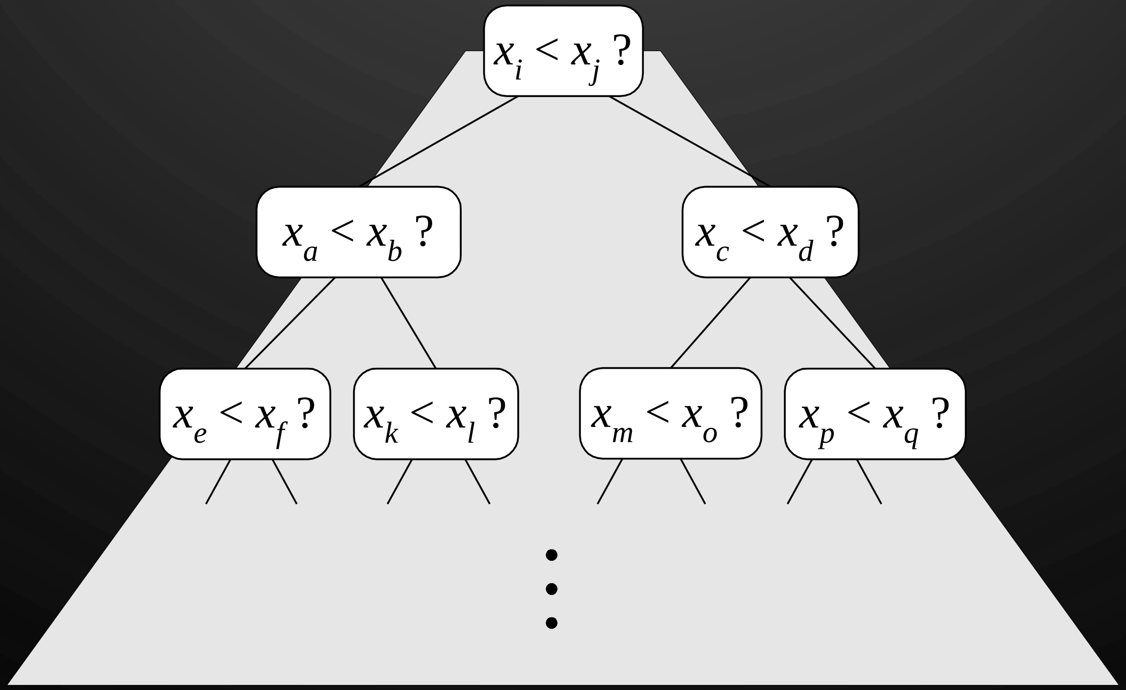
COMPARISON-BASED SORTING

- Many sorting algorithms are comparison based.
  - They sort by making comparisons between pairs of objects
  - Examples: bubble-sort, selection-sort, insertion-sort, heap-sort, merge-sort, quick-sort, ...
- Let us therefore derive a lower bound on the running time of any algorithm that uses comparisons to sort n elements, $x_1, x_2, \ldots, x_n$.



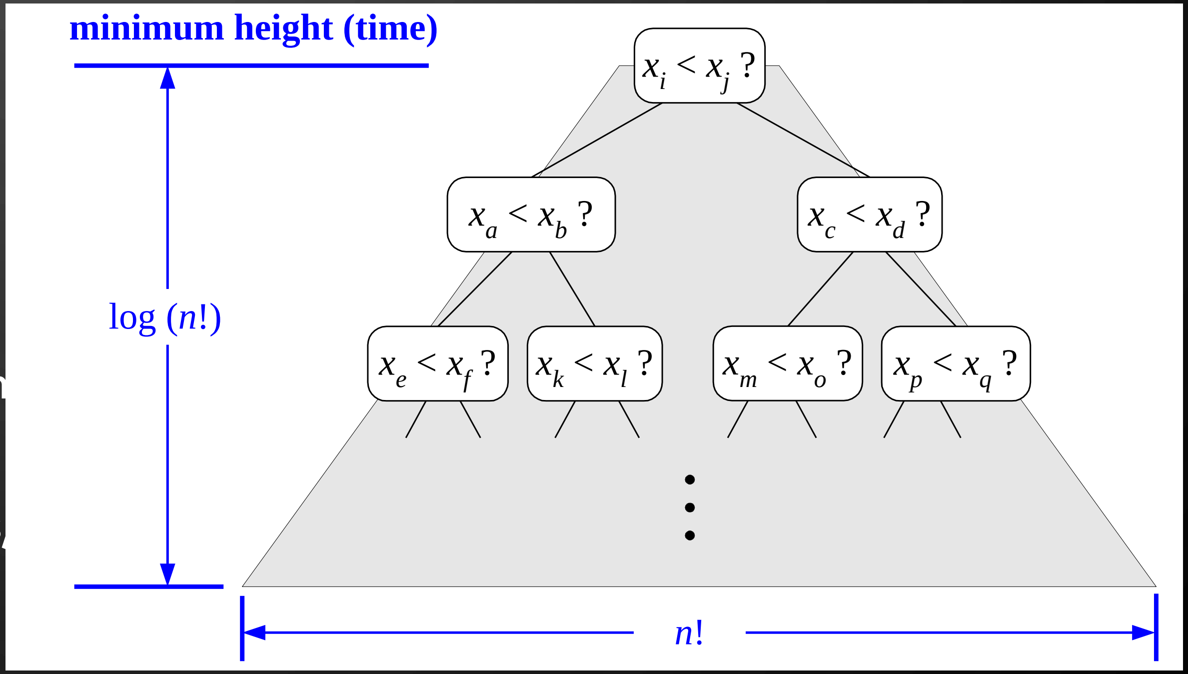
COUNTING COMPARISONS

- Let us just count comparisons then.
- Each possible run of the algorithm corresponds to a root-to-leaf path in a decision tree

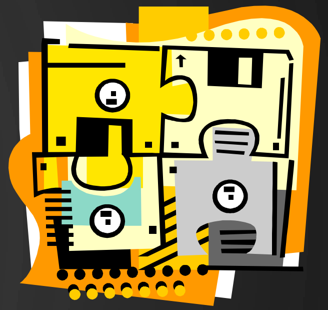


DECISION TREE HEIGHT

- The height of the decision tree is a lower bound on the running time
- Every input permutation must lead to a separate leaf output
- If not, some input …4…5… would have same output ordering as …5…4…, which would be wrong
- Since there are $n! = 1 * 2 * \cdots * n$ leaves, the height is at least $\log(n!)$



THE LOWER BOUND



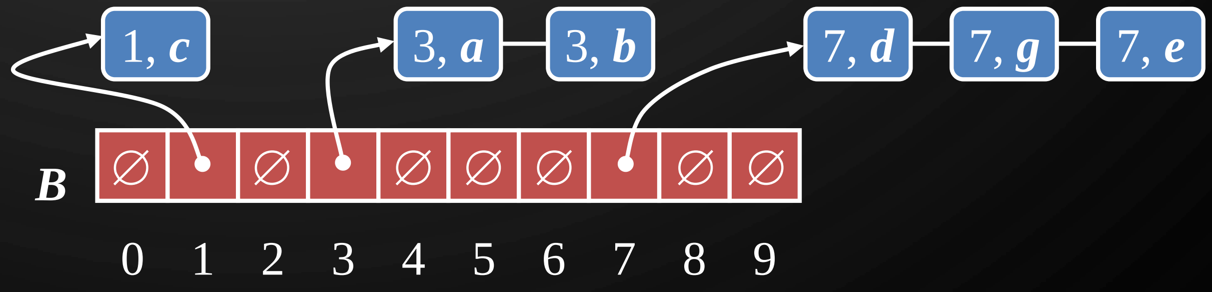
- Any comparison-based sorting algorithm takes at least $\log(n!)$ time

$$\log(n!) \geq \log\left(\frac{n}{2}\right)^{\frac{n}{2}} = \frac{n}{2}\log\frac{n}{2}$$

- That is, any comparison-based sorting algorithm must run in $\Omega(n \log n)$ time.

BUCKET-SORT AND RADIX-SORT

CAN WE SORT IN LINEAR TIME?



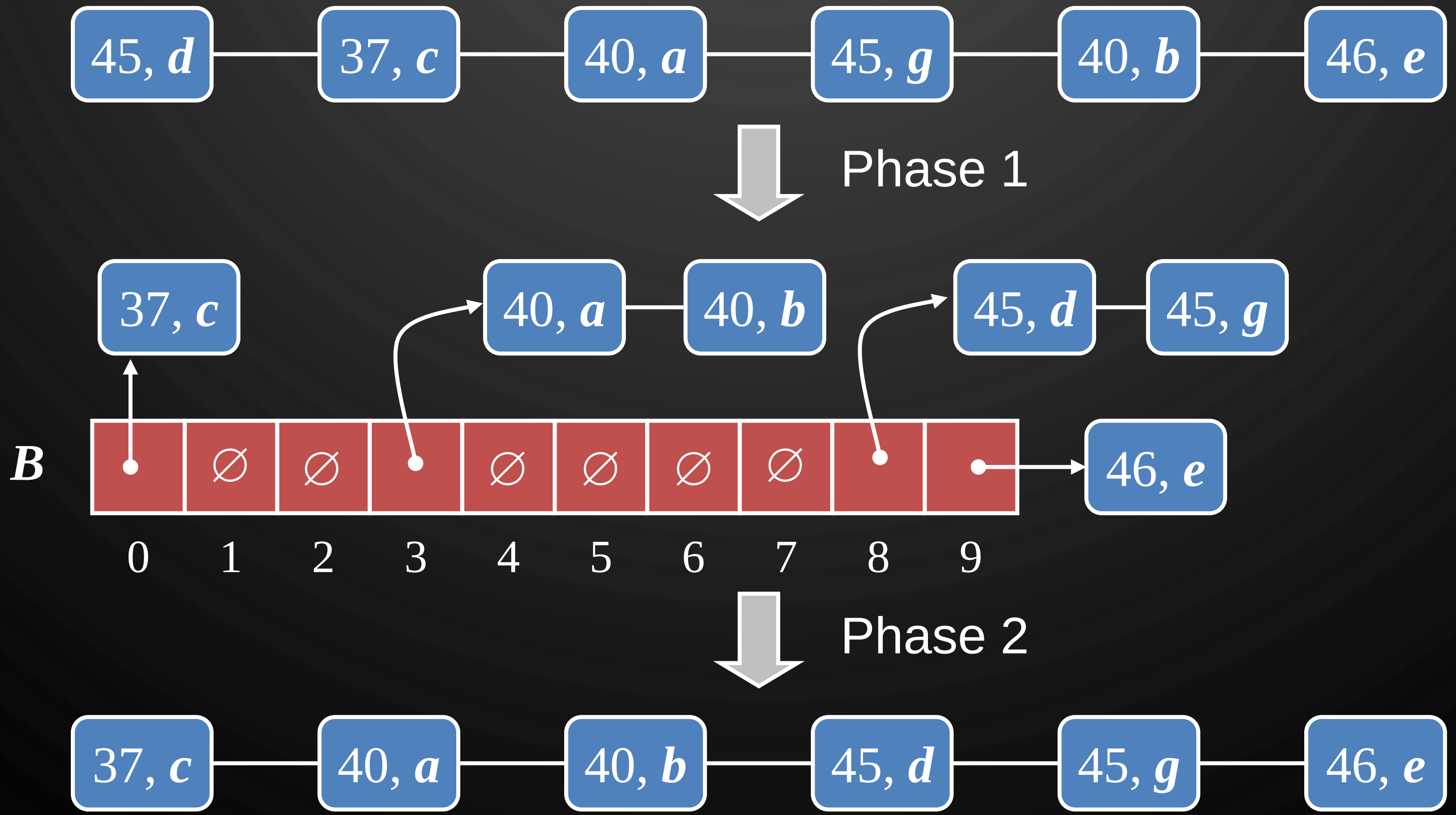
BUCKET-SORT

- Let be S be a sequence of n (key, element) items with keys in the range $[0, N-1]$
- **Bucket-sort** uses the keys as indices into an auxiliary array B of sequences (buckets)
  - Phase 1: Empty sequence S by moving each entry into its bucket $B[k]$
  - Phase 2: for $i \leftarrow 0 \ldots N-1$, move the items of bucket $B[i]$ to the end of sequence S
- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n+N)$ time
- Bucket-sort takes $O(n+N)$ time

```
Algorithm bucketSort(S, N)
Input: Sequence S of entries with
       integer keys in the range [0, N − 1]
Output: Sequence S sorted in
nondecreasing
        order of the keys
B ← array of N empty sequences
for each entry e ∈ S do
  k ← e.key()
  remove e from S
  insert e at the end of bucket B[k]
for i ← 0 to N − 1 do
  for each entry e ∈ B[i] do
    remove e from bucket B[i]
    insert e at the end of S
```

EXAMPLE

- Key range [37, 46] – map to buckets [0,9]



PROPERTIES AND EXTENSIONS

- Properties
  - Key-type
    - The keys are used as indices into an array and cannot be arbitrary objects
    - **No external comparator**
  - Stable sorting
    - The relative order of any two items with the same key is preserved after the execution of the algorithm
- Extensions
  - Integer keys in the range $[a, b]$
    - Put entry e into bucket $B[k - a]$
  - String keys from a set D of possible strings, where D has constant size (e.g., names of the 50 U.S. states)
    - Sort D and compute the index $i(k)$ of each string k of D in the sorted sequence
    - Put item e into bucket $B[i(k)]$

LEXICOGRAPHIC ORDER

- Given a list of tuples:

  (7,4,6) (5,1,5) (2,4,6) (2,1,4) (5,1,6) (3,2,4)

- After sorting, the list is in lexicographical order:

  (2,1,4) (2,4,6) (3,2,4) (5,1,5) (5,1,6) (7,4,6)

LEXICOGRAPHIC ORDER FORMALIZED

- A d-tuple is a sequence of d keys $(k_1, k_2, \dots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
  - Example - the Cartesian coordinates of a point in space is a 3-tuple (x, y, z)
- The lexicographic order of two d-tuples is recursively defined as follows
- $(x_1, x_2, \dots, x_d) < (y_1, y_2, \dots, y_d) \Leftrightarrow$
$$x_1 < y_1 \vee \big(x_1 = y_1 \wedge (x_2, \dots, x_d) < (y_2, \dots, y_d)\big)$$
- i.e., the tuples are compared by the first dimension, then by the second dimension, etc.

EXERCISE
LEXICOGRAPHIC ORDER

- Given a list of 2-tuples, we can order the tuples lexicographically by applying a stable sorting algorithm two times:
  (3,3) (1,5) (2,5) (1,2) (2,3) (1,7) (3,2) (2,2)
- Possible ways of doing it:
  - Sort first by 1st element of tuple and then by 2nd element of tuple
  - Sort first by 2nd element of tuple and then by 1st element of tuple
- Show the result of sorting the list using both options

EXERCISE
LEXICOGRAPHIC ORDER

- (3,3) (1,5) (2,5) (1,2) (2,3) (1,7) (3,2) (2,2)
- Using a stable sort,
  - Sort first by 1st element of tuple and then by 2nd element of tuple
  - Sort first by 2nd element of tuple and then by 1st element of tuple
- Option 1:
  - 1st sort: (1,5) (1,2) (1,7) (2,5) (2,3) (2,2) (3,3) (3,2)
  - 2nd sort: (1,2) (2,2) (3,2) (2,3) (3,3) (1,5) (2,5) (1,7) - WRONG
- Option 2:
  - 1st sort: (1,2) (3,2) (2,2) (3,3) (2,3) (1,5) (2,5) (1,7)
  - 2nd sort: (1,2) (1,5) (1,7) (2,2) (2,3) (2,5) (3,2) (3,3) - CORRECT

LEXICOGRAPHIC-SORT

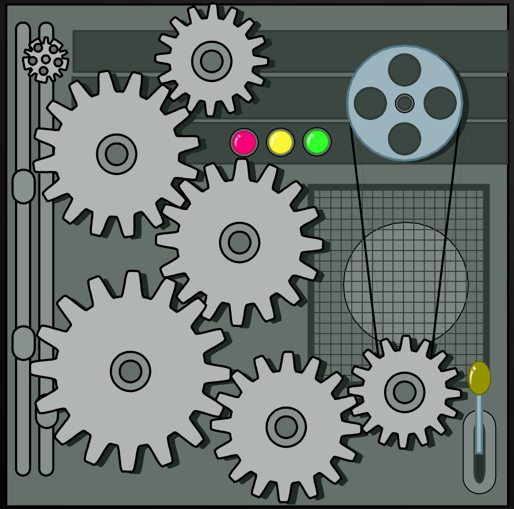
- Let C_i be the comparator that compares two tuples by their i-th dimension
- Let `stableSort`(S, C) be a stable sorting algorithm that uses comparator C
- Lexicographic-sort sorts a sequence of d-tuples in lexicographic order by executing d times algorithm `stableSort`, one per dimension
- Lexicographic-sort runs in $O\big(dT(n)\big)$ time, where $T(n)$ is the running time of `stableSort`

Algorithm lexicographicSort(S)
Input: Sequence S of d-tuples
Output: Sequence S sorted in lexicographic order

1. **for** $i \leftarrow d$ **to** 1 **do**
2. stableSort(S, C_i)

RADIX-SORT

- Radix-sort is a specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, N-1]$
- Radix-sort runs in time $O\big(d(n+N)\big)$

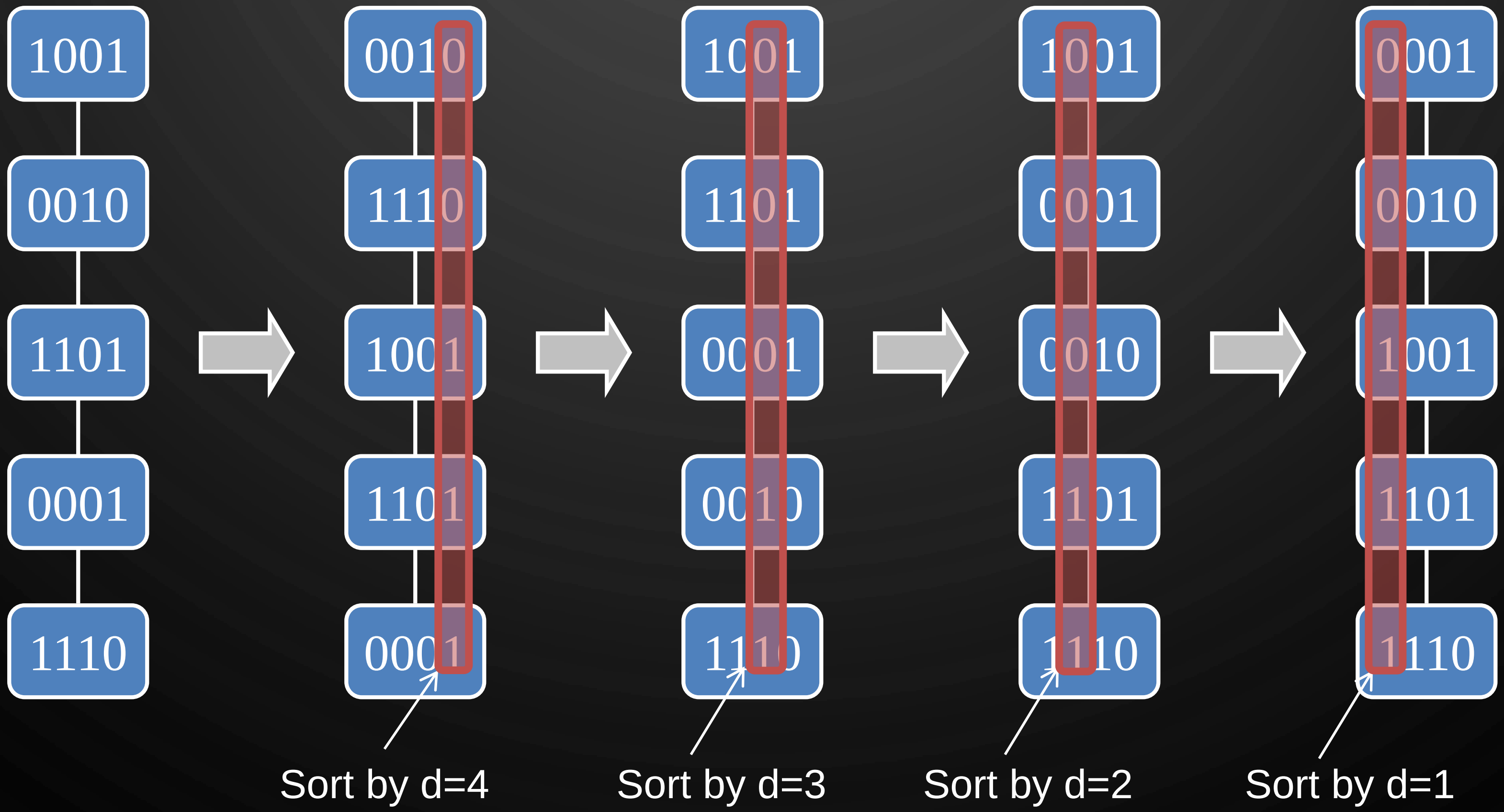


Algorithm radixSort(S, N)
Input: Sequence S of d-tuples such that
$(0, \dots, 0) \le (x_1, \dots, x_d)$ and
$(x_1, \dots, x_d) \le (N-1, \dots, N-1)$
for each tuple $(x_1, \dots, x_d)$ in S
Output: Sequence S sorted in lexicographic order

for $i \leftarrow d$ **to** 1 **do**
    set the key k of each entry $\big(k, (x_1, \dots, x_d)\big)$ of S to ith dimension x_i
    bucketSort(S, N)

EXAMPLE
RADIX-SORT FOR BINARY NUMBERS

- Sorting a sequence of 4-bit integers
  - $d = 4, N = 2$ so $O\big(d(n+N)\big) = O\big(4(n+2)\big) = O(n)$



SUMMARY OF SORTING ALGORITHMS

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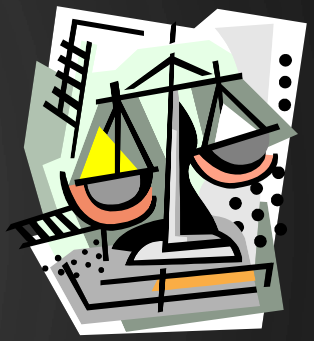
Heap Sort	$O(n \log n)$	In-place Fast, for large data sets
Quick Sort	Exp. $O(n \log n)$ AC, BC $O(n^2)$ WC	Randomized, in-place Fastest, for large data sets
Merge Sort	$O(n \log n)$	Sequential data access Fast, for huge data sets
Radix Sort	$O\big(d(n+N)\big)$, d #digits, N range of digit values	Stable Fastest, only for integers

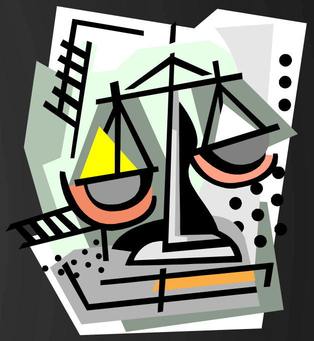
SELECTION

THE SELECTION PROBLEM



- Given an integer k and n elements $\{x_1, x_2, \ldots, x_n\}$, taken from a total order, find the k-th smallest element in this set.
    - Also called order statistics, the ith order statistic is the ith smallest element
    - Minimum - $k = 1$ - 1st order statistic
    - Maximum - $k = n$ - nth order statistic
    - Median - $k = \left\lfloor \frac{n}{2} \right\rfloor$
    - etc

THE SELECTION PROBLEM

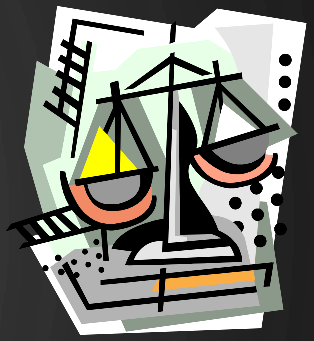


- Naïve solution - SORT!
- We can sort the set in $O(n \log n)$ time and then index the k-th element.

$7\ 4\ 9\ \underline{6}\ 2 \rightarrow 2\ 4\ \underline{6}\ 7\ 9$ k=3

- Can we solve the selection problem faster?

THE MINIMUM (OR MAXIMUM)

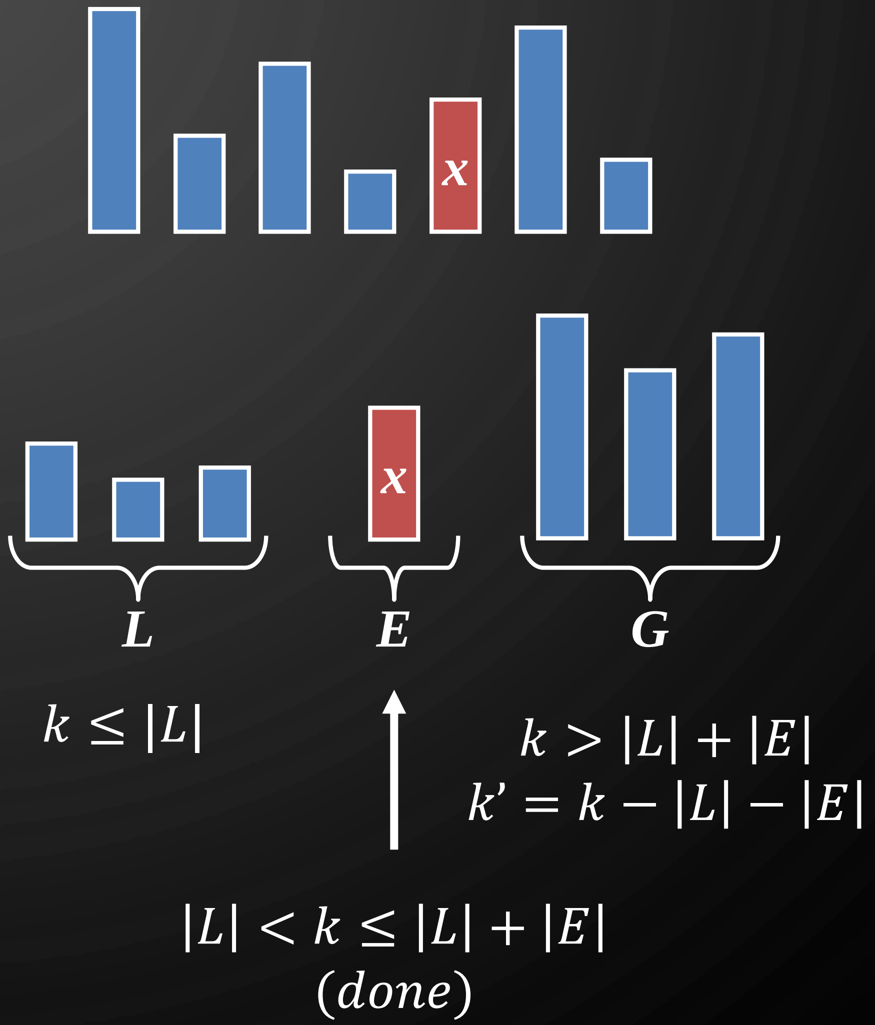


Algorithm minimum(A)
Input: Array A
Output: minimum element in A
1. $m \leftarrow A[1]$
2. **for** $i \leftarrow 2$ **to** n **do**
3. $\quad m \leftarrow \min(m, A[i])$
4. **return** m

- Running Time
  - $O(n)$
- Is this the best possible?

QUICK-SELECT

- **Quick-select** is a randomized selection algorithm based on the prune-and-search paradigm:
  - Prune: pick a random element x (called pivot) and partition S into
    - L elements $< x$
    - E elements $= x$
    - G elements $> x$
  - Search: depending on k, either answer is in E, or we need to recur on either L or G
- Note: Partition same as Quicksort



QUICK-SELECT VISUALIZATION

- An execution of quick-select can be visualized by a recursion path
  - Each node represents a recursive call of quick-select, and stores k and the remaining sequence

$$k = 5, S = (7, 4, 9, \underline{3}, 2, 6, 5, 1, 8)$$

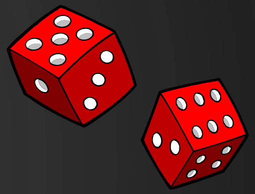
$$k = 2, S = (7, 4, 9, 6, 5, \underline{8})$$

$$k = 2, S = (7, \underline{4}, 6, 5)$$

$$k = 1, S = (7, 6, \underline{5})$$

$$5$$

EXERCISE



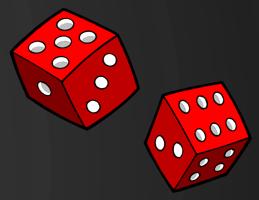
- Best Case - even splits (n/2 and n/2)
- Worst Case - bad splits (1 and n-1)

Good call

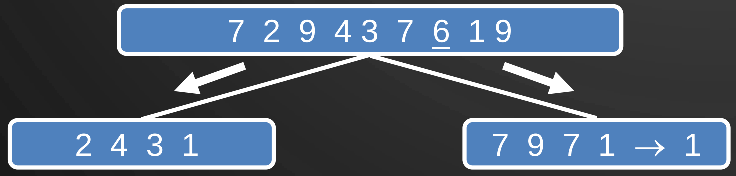
Bad call

- Derive and solve the recurrence relation corresponding to the best case performance of randomized quick-select.
- Derive and solve the recurrence relation corresponding to the worst case performance of randomized quick-select.

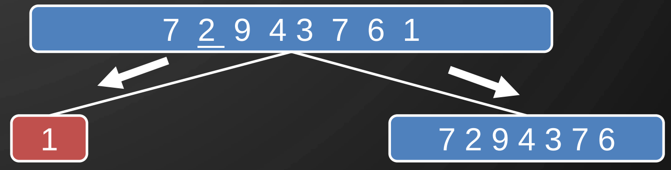
EXPECTED RUNNING TIME



- Consider a recursive call of quick-select on a sequence of size s
  - Good call: the size of L and G is at most $\frac{3s}{4}$
  - Bad call: the size of L and G is greater than $\frac{3s}{4}$

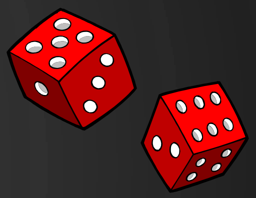


Good call



Bad call

- A call is good with probability $1/2$
  - $1/2$ of the possible pivots cause good calls:

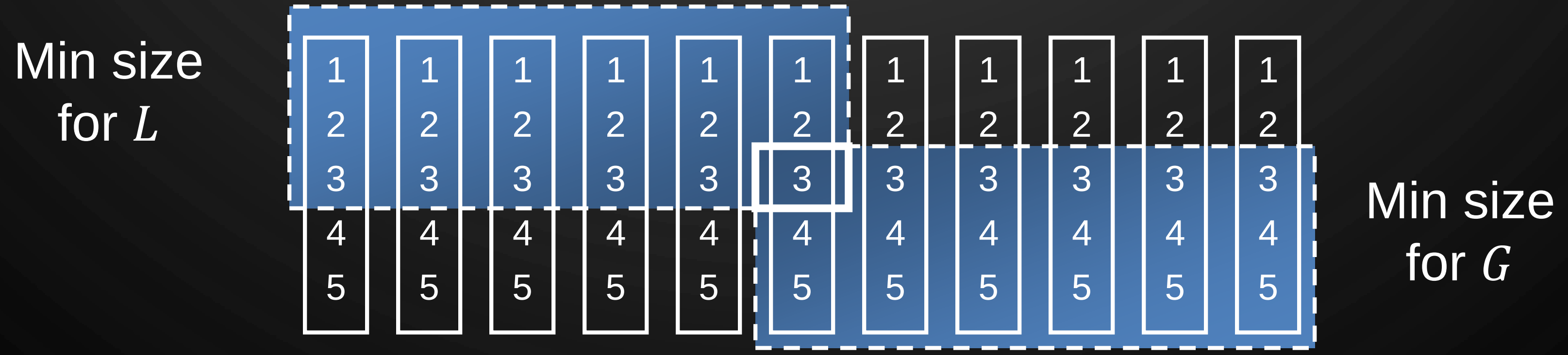


EXPECTED RUNNING TIME

- Probabilistic Fact #1: The expected number of coin tosses required in order to get one head is two
- Probabilistic Fact #2: Expectation is a linear function:
  - $E(X+Y) = E(X) + E(Y)$
  - $E(cX) = cE(X)$
- Let $T(n)$ denote the expected running time of quick-select.
- By Fact #2, $T(n) < T\left(\frac{3n}{4}\right) + bn * (expected\ \#\ of\ calls\ before\ a\ good\ call)$
- By Fact #1, $T(n) < T\left(\frac{3n}{4}\right) + 2bn$
- That is, $T(n)$ is a geometric series: $T(n) < 2bn + 2b\left(\frac{3}{4}\right)n + 2b\left(\frac{3}{4}\right)^2 n + 2b\left(\frac{3}{4}\right)^3 n + \cdots$
- So $T(n)$ is $O(n)$.
- We can solve the selection problem in $O(n)$ expected time.

DETERMINISTIC SELECTION

- We can do selection in $O(n)$ worst-case time.
- Main idea: recursively use the selection algorithm itself to find a good pivot for quick-select:
  - Divide S into $\frac{n}{5}$ sets of 5 each
  - Find a median in each set
  - Recursively find the median of the “baby” medians.



- See Exercise C-12.56 for details of analysis.

INTERVIEW QUESTION 1

- You are given two sorted arrays, A and B, where A has a large enough buffer at the end to hold B. Write a method to merge B into A in sorted order.

GAYLE LAAKMANN MCDOWELL, "CRACKING THE CODE INTERVIEW: 150 PROGRAMMING QUESTIONS AND SOLUTIONS", 5TH EDITION, CAREERCUP PUBLISHING, 2011.

INTERVIEW QUESTION 2

- Write a method to sort an array of strings so that all the anagrams are next to each other.
  - Two words are anagrams if they use the exact same letters, i.e., race and care are anagrams

GAYLE LAAKMANN MCDOWELL, "CRACKING THE CODE INTERVIEW: 150 PROGRAMMING QUESTIONS AND SOLUTIONS", 5TH EDITION, CAREERCUP PUBLISHING, 2011.

INTERVIEW QUESTION 3

- Imagine you have a 2 TB file with one string per line. Explain how you would sort the file.

GAYLE LAAKMANN MCDOWELL, "CRACKING THE CODE INTERVIEW: 150 PROGRAMMING QUESTIONS AND SOLUTIONS", 5TH EDITION, CAREERCUP PUBLISHING, 2011.