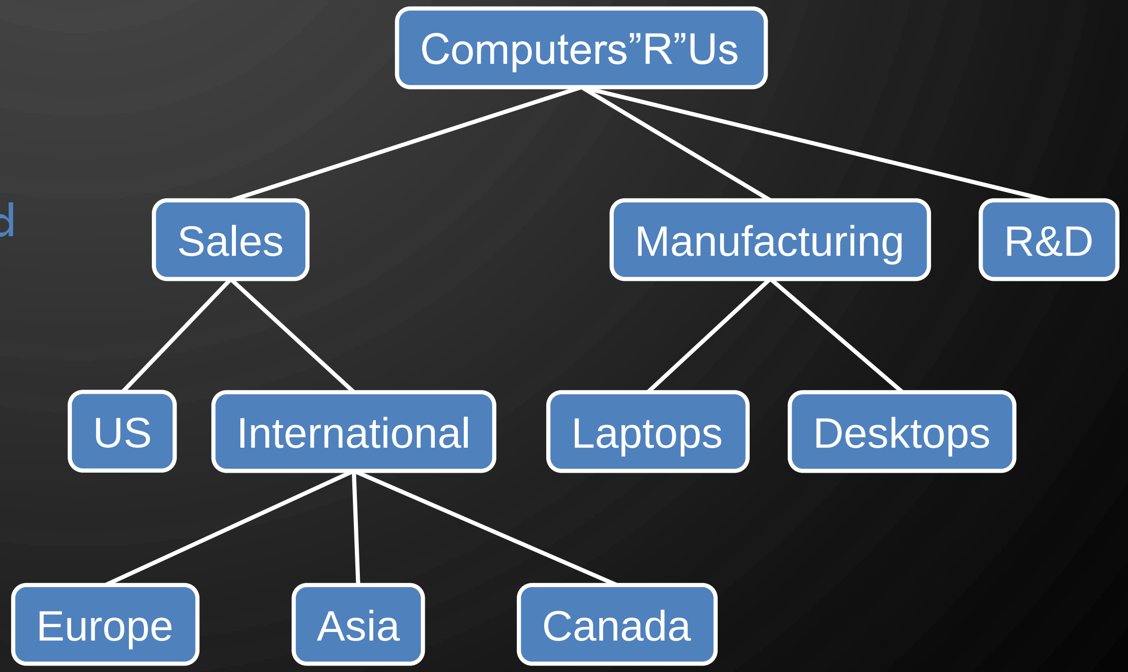
CH8
TREES

ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN JAVA, GOODRICH, TAMASSIA AND GOLDWASSER (WILEY 2016)

WHAT IS A TREE

- In computer science, a tree is an abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation
- Applications:
  - Organization charts
  - File systems
  - Programming environments

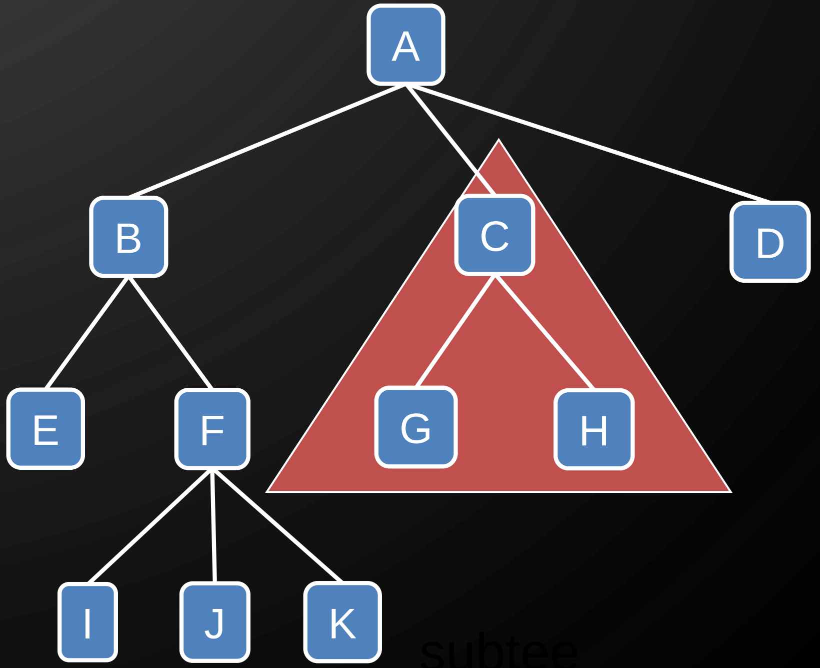


FORMAL DEFINITION

- A tree T is a set of nodes storing elements in a parent-child relationship with the following properties:
  - If T is nonempty, it has a special node called the **root** of T, that has no parent
  - Each node v of T different from the root has a unique **parent** node w; every node with parent w is a **child** of w
- Note that trees can be empty and can be defined recursively!
- Note each node can have zero or more children

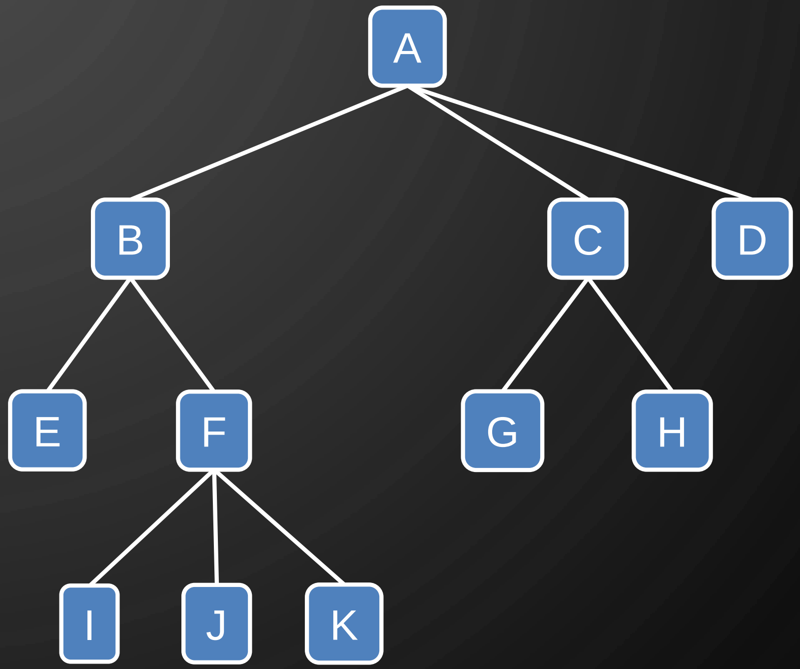
TREE TERMINOLOGY

- Root: node without parent (A)
- Internal node: node with at least one child (A, B, C, F)
- Leaf (aka External node): node without children (E, I, J, K, G, H, D)
- Ancestors of a node: parent, grandparent, great-grandparent, etc.
- Siblings of a node: Any node which shares a parent
- Depth of a node: number of ancestors
- Height of a tree: maximum depth of any node (3)
- Descendant of a node: child, grandchild, great-grandchild, etc.
- Subtree: tree consisting of a node and its descendants
- Edge: a pair of nodes (u, v) such that u is a parent of v $((C, H))$
- Path: A sequence of nodes such that any two consecutives nodes form an edge(A, B, F, J)
- A tree is ordered when there is a linear ordering defined for the children of each node



EXERCISE

- Answer the following questions about the tree shown on the right:
  - What is the size of the tree (number of nodes)?
  - Classify each node of the tree as a root, leaf, or internal node
  - List the ancestors of nodes B, F, G, and A. Which are the parents?
  - List the descendants of nodes B, F, G, and A. Which are the children?
  - List the depths of nodes B, F, G, and A.
  - What is the height of the tree?
  - Draw the subtrees that are rooted at node F and at node K.

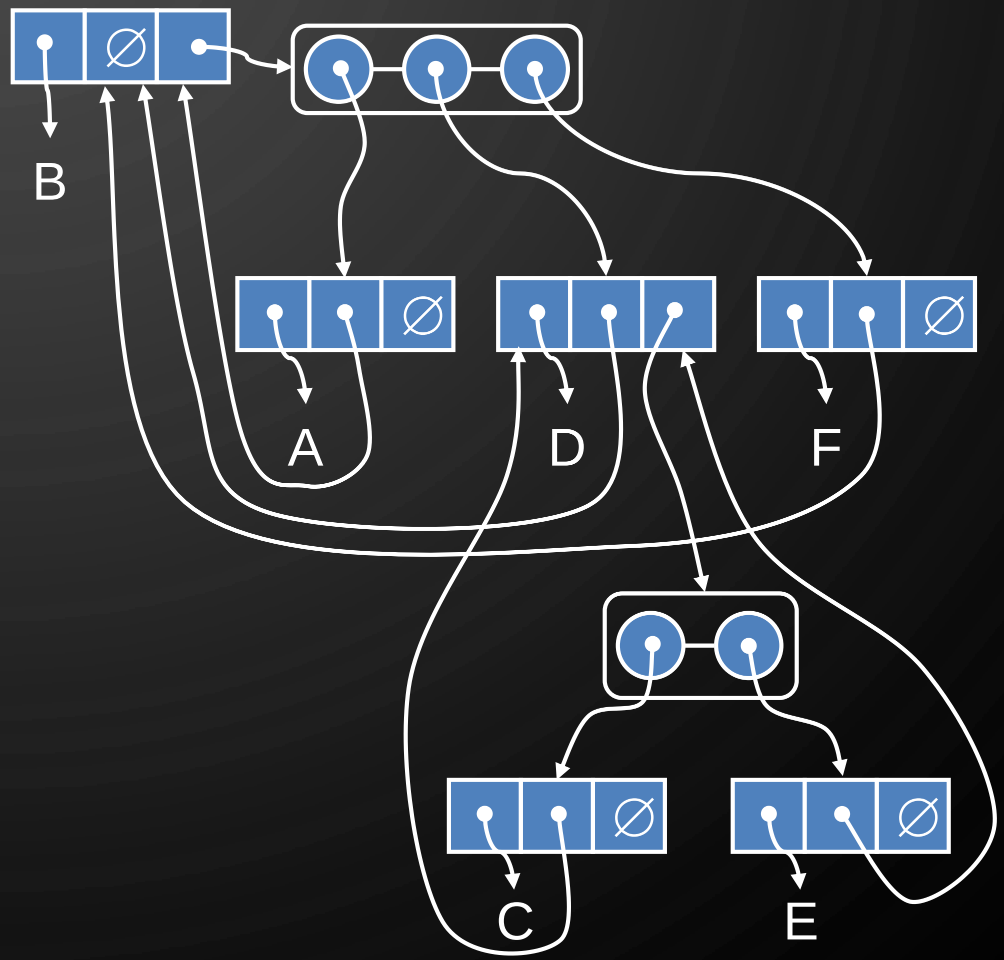


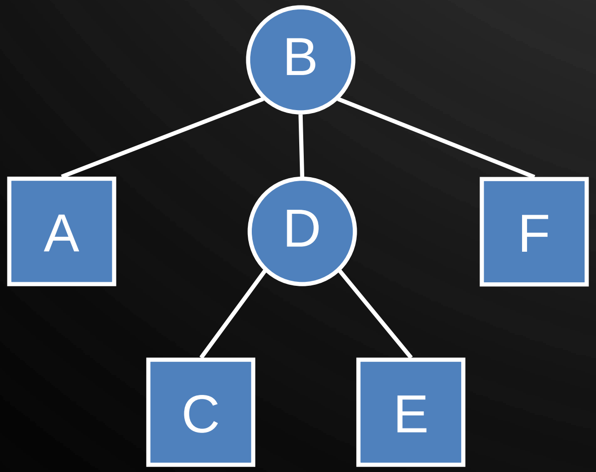
TREE ADT

- We use positions to abstract nodes as we don't want to expose the internals of our implementation
- Generic methods:
  - `Integer size()`
  - `boolean isEmpty()`
  - `Iterator iterator()`
  - `Iterable positions()`
- Accessor methods:
  - `Position root()`
  - `Position parent(p)`
  - `Iterable children(p)`
  - `Integer numChildren(p)`
- Query methods:
  - `Boolean isInternal(p)`
  - `Boolean isExternal(p)`
  - `Boolean isRoot(p)`
- Additional update methods may be defined by data structures implementing the Tree ADT

A LINKED STRUCTURE FOR GENERAL TREES

- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes
- Node objects implement the Position ADT





EXAMPLE NODE CLASS FOR GENERAL TREE

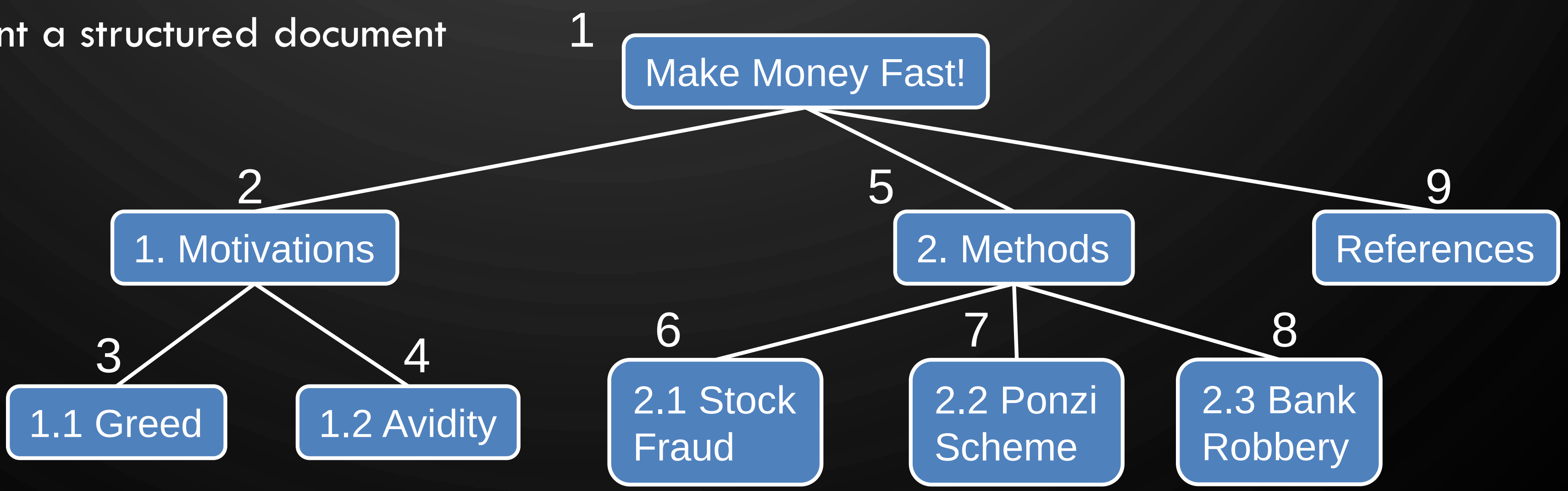
```
public class GeneralTreeNode<ElementType>
    implements Position<ElementType> {
  ElementType element;
  GeneralTreeNode<ElementType> parent;
  ArrayList<GeneralTreeNode<ElementType>> children;
  // … Constructors, accessors, setters
}
```

PREORDER TRAVERSAL

- A **traversal** visits the nodes of a tree in a systematic manner
- In a **preorder traversal**, a node is visited before its descendants
- Application: print a structured document

```
Algorithm preOrder
Input: Tree T
preOrder(T, T.root())

Algorithm preOrder
Input: Tree T, Position p
visit-action(p)
for each Position c ∈ T.children(p) do
    preOrder(T, c)
```

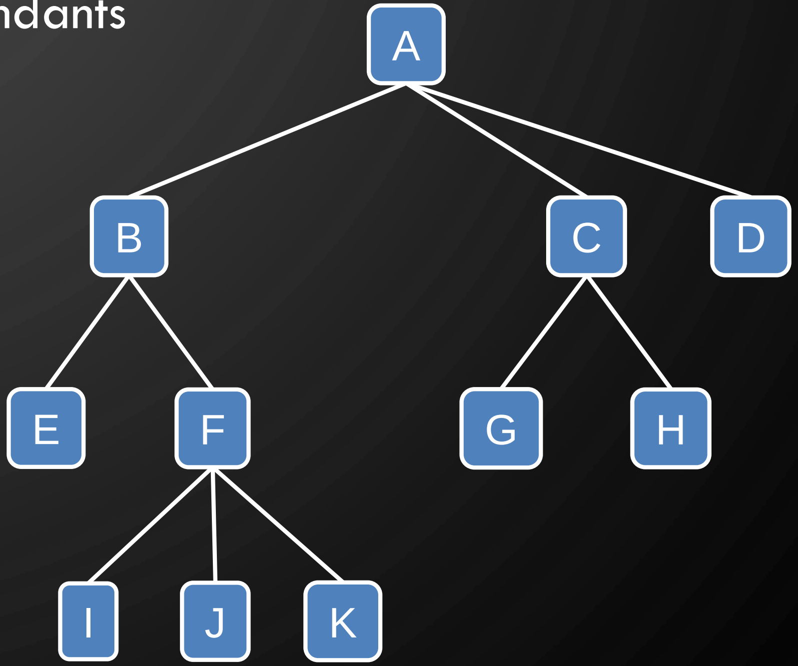


EXERCISE: PREORDER TRAVERSAL

- In a **preorder traversal**, a node is visited before its descendants
- List the nodes of this tree in preorder traversal order.

```
Algorithm preOrder
Input: Tree T
preOrder(T, T.root())

Algorithm preOrder
Input: Tree T, Position p
visit-action(p)
for each Position c ∈ T.children(p) do
    preOrder(T, c)
```

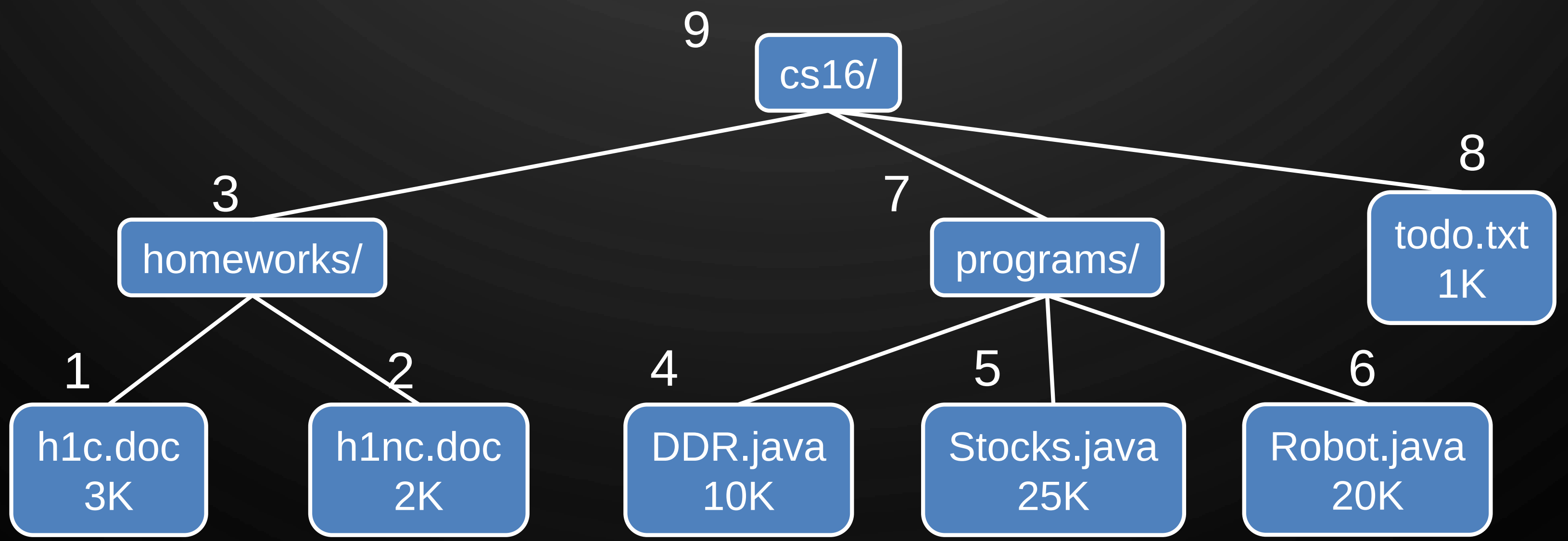


POSTORDER TRAVERSAL

- In a **postorder traversal**, *a node is visited after its* descendants
- Application: compute space used by files in a directory and its subdirectories

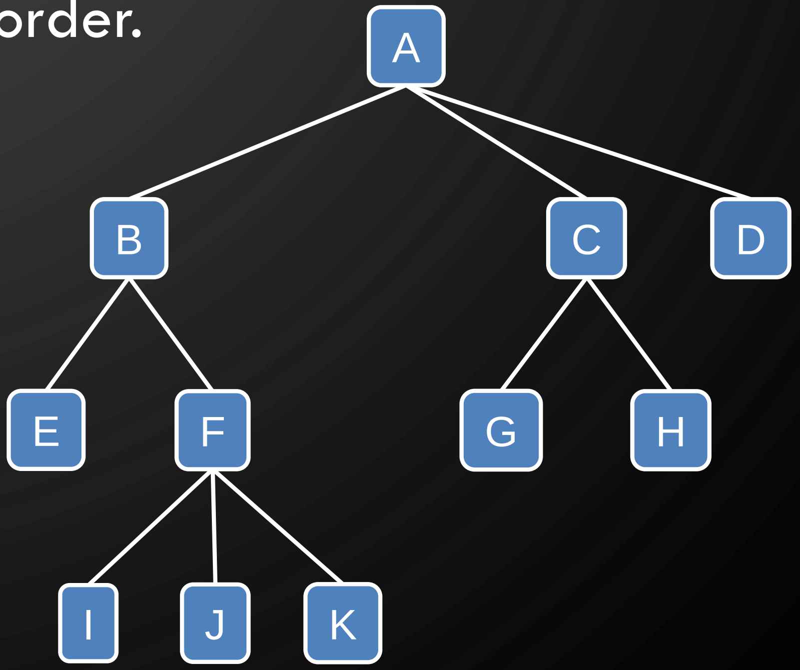
```
Algorithm postOrder
Input: Tree T
postOrder(T, T.root())
```

```
Algorithm postOrder
Input: Tree T, Position p
for each Position c ∈ T.children(p) do
    postOrder(T, c)
visit-action(p)
```



EXERCISE: POSTORDER TRAVERSAL

- In a **postorder traversal**, a node is visited after its descendants
- List the nodes of this tree in postorder traversal order.



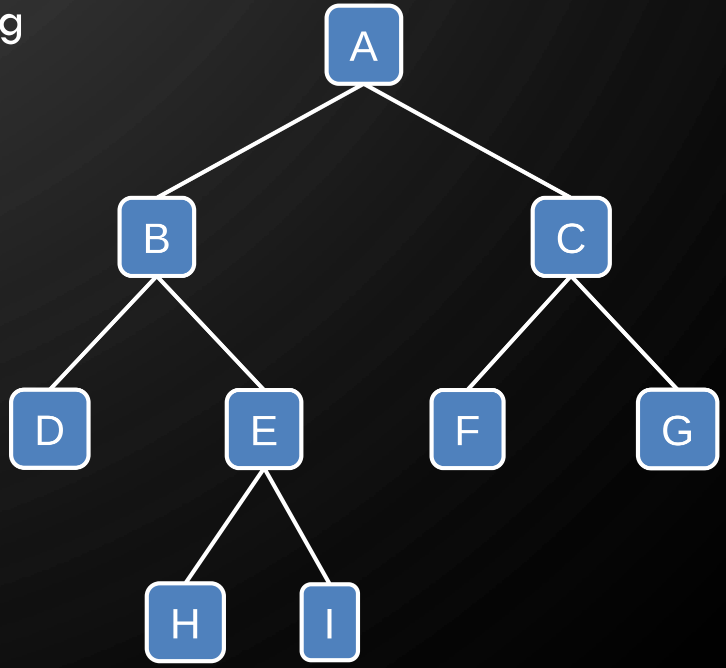
Algorithm **postOrder**
Input: Tree T
1. postOrder(T, T.root())

Algorithm **postOrder**
Input: Tree T, **Position** p
1. **for each Position** $c \in T$.children(p) **do**
2. postOrder(T, c)
3. visit-action(p)

BINARY TREE

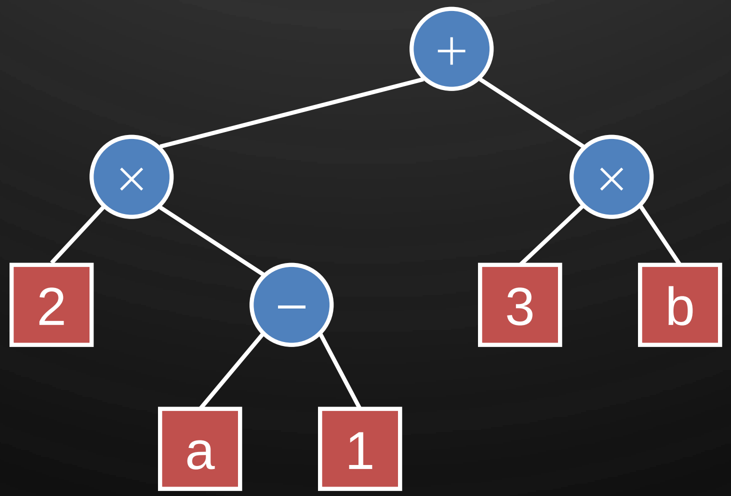
- A **binary tree** is a tree with the following properties:
  - Each internal node has two children
  - The children of a node are an ordered pair
- We call the children of an internal node **left child** and **right child**
- If a child has only one child, the tree is **improper**
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a binary tree

- Applications
  - Arithmetic expressions
  - Decision processes
  - Searching



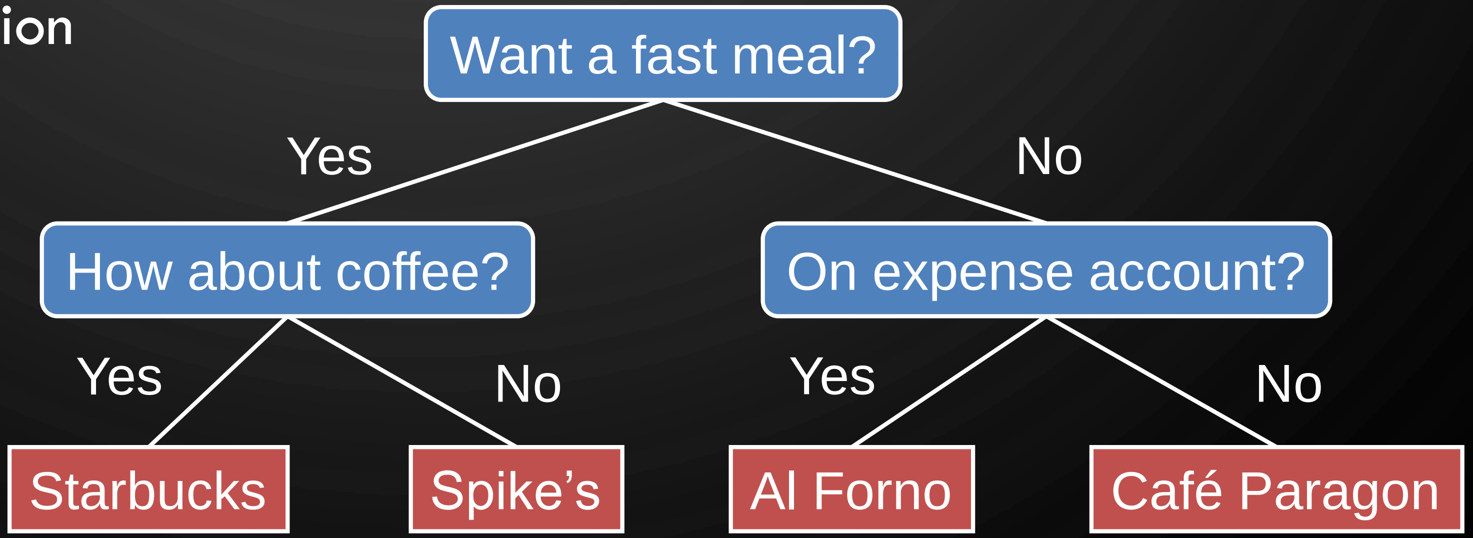
ARITHMETIC EXPRESSION TREE

- Binary tree associated with an arithmetic expression
  - Internal nodes: operators
  - Leaves: operands
- Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$



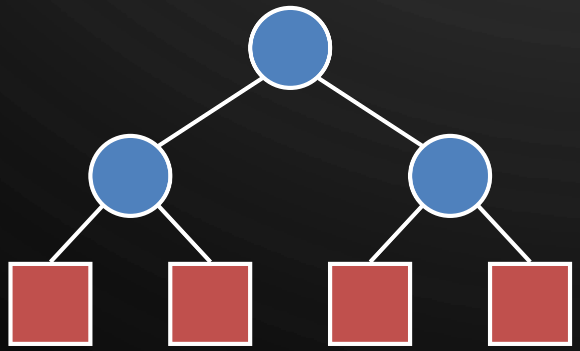
DECISION TREE

- Binary tree associated with a decision process
  - Internal nodes: questions with yes/no answer
  - Leaves: decisions
- Example: dining decision

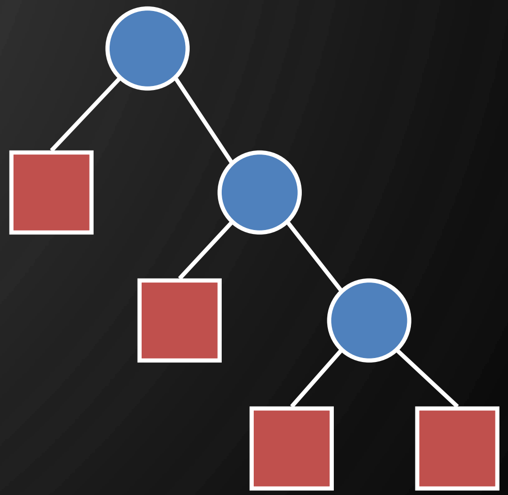


PROPERTIES OF BINARY TREES

- Notation
  - n number of nodes
  - e number of external nodes
  - i number of internal nodes
  - h height



- Properties:
  - $e = i + 1$
  - $n = 2e - 1$
  - $h \leq i$
  - $h \leq \frac{n-1}{2}$
  - $e \leq 2^h$
  - $h \geq \log_2 e$
  - $h \geq \log_2(n + 1) - 1$

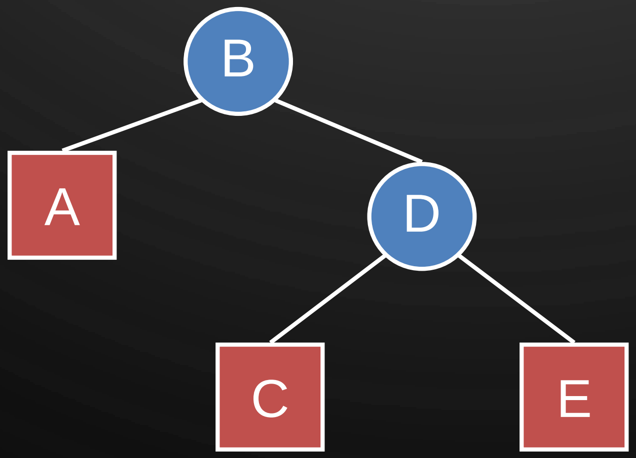


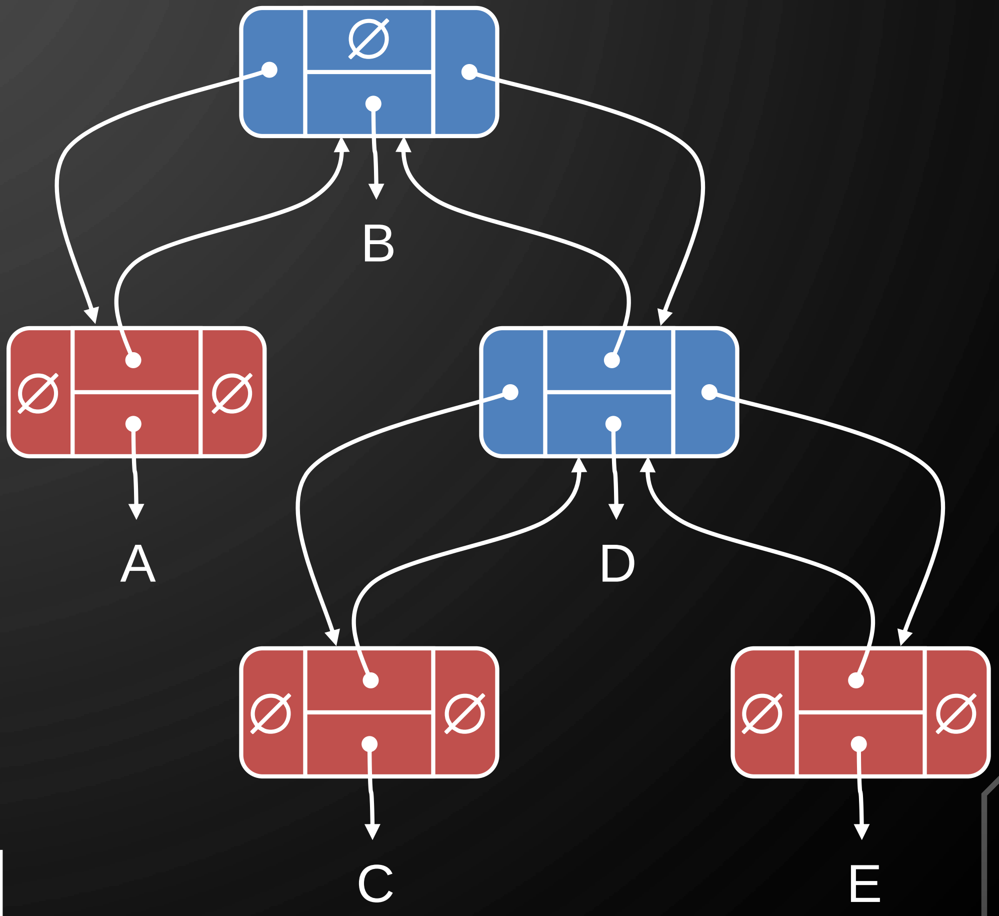
BINARY TREE ADT

- The Binary Tree ADT extends the Tree ADT, i.e., it inherits all the methods of the Tree ADT
- Additional position methods:
  - `Position left(p)`
  - `Position right(p)`
  - `Position sibling(p)`
- The above methods return null when there is no left, right, or sibling of p, respectively
- Update methods may also be defined by data structures implementing the Binary Tree ADT

A LINKED STRUCTURE FOR BINARY TREES

- A node is represented by an object storing
  - Element
  - Parent node
  - Left child node
  - Right child node





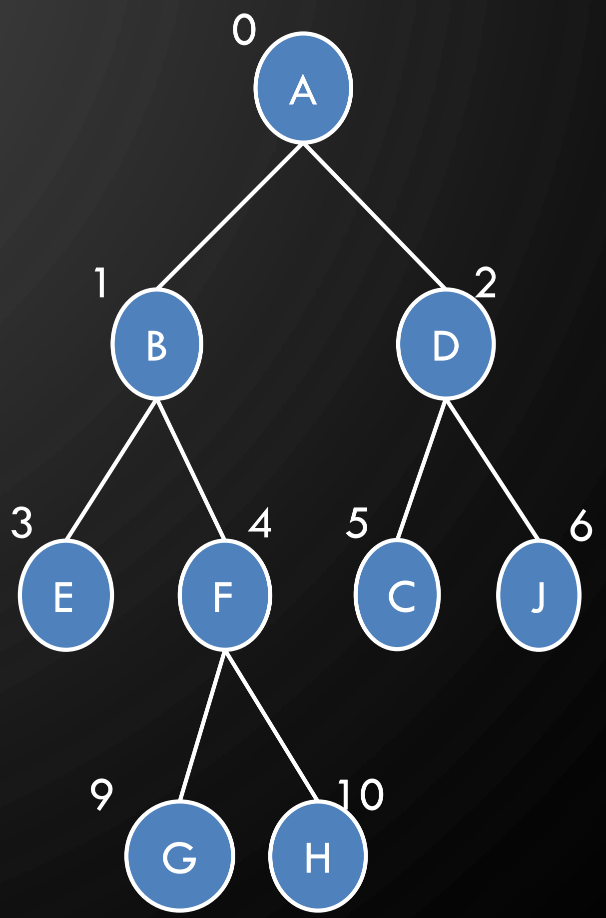
EXAMPLE NODE CLASS FOR BINARY TREE

```
public class BinaryTreeNode<ElementType>
    implements Position<ElementType> {
  ElementType element;
  BinaryTreeNode<ElementType> parent, left, right;
  // … Constructors, accessors, setters
}
```

ARRAY-BASED REPRESENTATION OF BINARY TREES

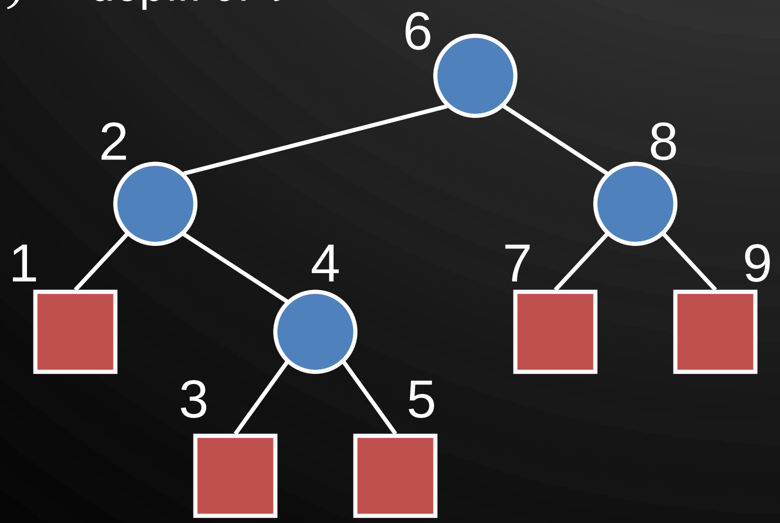
- Nodes are stored in an array A

- Node v is stored at $A[rank(V)]$
  - `rank(root) = 0`
  - if node is the left child of parent(node),
    `rank(node) = 2 * rank(parent(node)) + 1`
  - if node is the right child of parent(node),
    `rank(node) = 2 * rank(parent(node)) + 2`



INORDER TRAVERSAL

- In an **inorder traversal** a node is visited after its left subtree and before its right subtree
- Application: draw a binary tree
  - $x(v)$ = inorder rank of v
  - $y(v)$ = depth of v

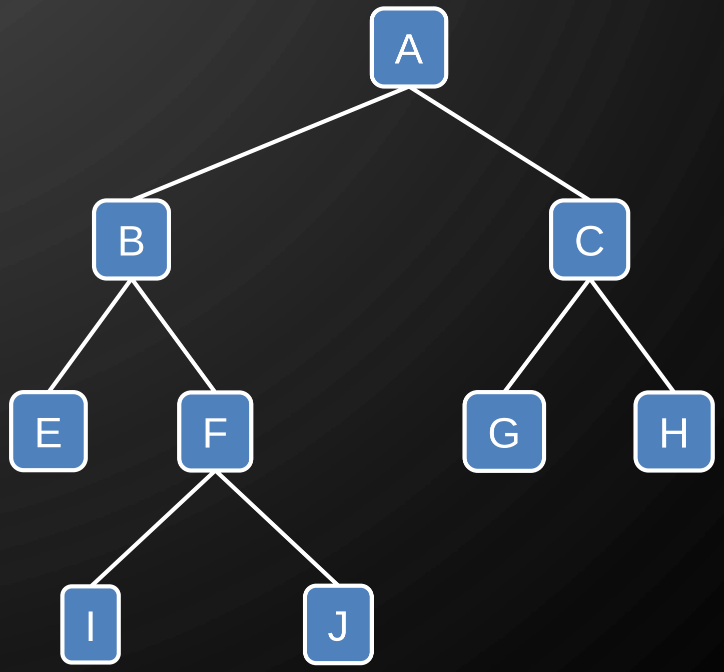



```
Algorithm inOrder
Input: Tree T
inOrder(T, T.root())
```

```
Algorithm inOrder
Input: Tree T, Position p
if T.left(p) ≠ null then
      inOrder(T, T.left(p))
visit-action(p)
if T.right(p) ≠ null then
      inOrder(T, T.right(p))
```

EXERCISE: INORDER TRAVERSAL

- In an **inorder traversal** a node is visited after its left subtree and before its right subtree
- List the nodes of this tree in inorder traversal order.




```
Algorithm inOrder
Input: Tree T
inOrder(T, T.root())
```

```
Algorithm inOrder
Input: Tree T, Position p
if T.left(p) ≠ null then
      inOrder(T, T.left(p))
visit-action(p)
if T.right(p) ≠ null then
      inOrder(T, T.right(p))
```

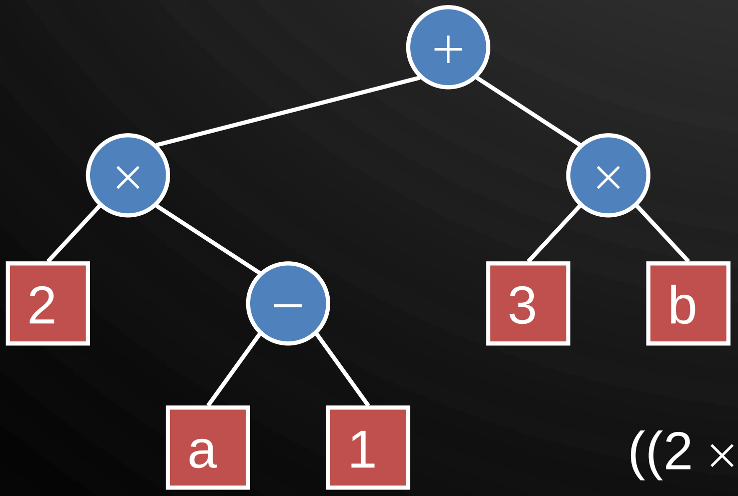
EXERCISE: PREORDER & INORDER TRAVERSAL

- Draw a (single) binary tree T, such that
  - Each internal node of T stores a single character
  - A preorder traversal of T yields EXAMFUN
  - An inorder traversal of T yields MAFXUEN

APPLICATION

PRINT ARITHMETIC EXPRESSIONS

- Specialization of an inorder traversal
  - print operand or operator when visiting node
  - print "(" before traversing left subtree
  - print ")" after traversing right subtree



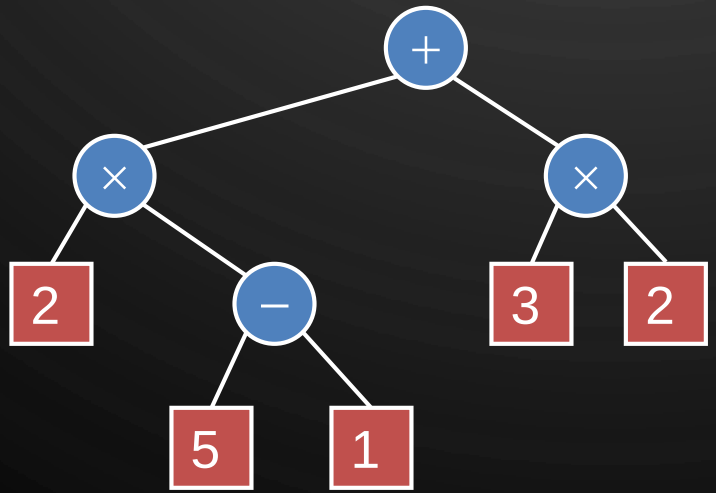
$((2 \times (a - 1)) + (3 \times b))$

```
Algorithm printExpr
Input: Tree T
1.printExpr(T, T.root())

Algorithm printExpr
Input: Tree T, Position p
1. if T.left(p) ≠ null then
2.      print("(")
3.      printExpr(T, T.left(p))
4. print(p.getElement())
5. if T.right(p) ≠ null then
6.      printExpr(T, T.right(p))
7.      print(")")
```

APPLICATION
EVALUATE ARITHMETIC EXPRESSIONS

- Specialization of a postorder traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees




```
Algorithm evalExpr
Input: Tree T
1.evalExpr(T, T.root())
```

```
Algorithm evalExpr
Input: Tree T, Position p
1. if T.isExternal(p) then
2.      return p.getElement()
3. x ←evalExpr(T, T.left(p))
4. y ←evalExpr(T, T.right(p))
5. ∘←operator stored at v
6. return x∘y
```

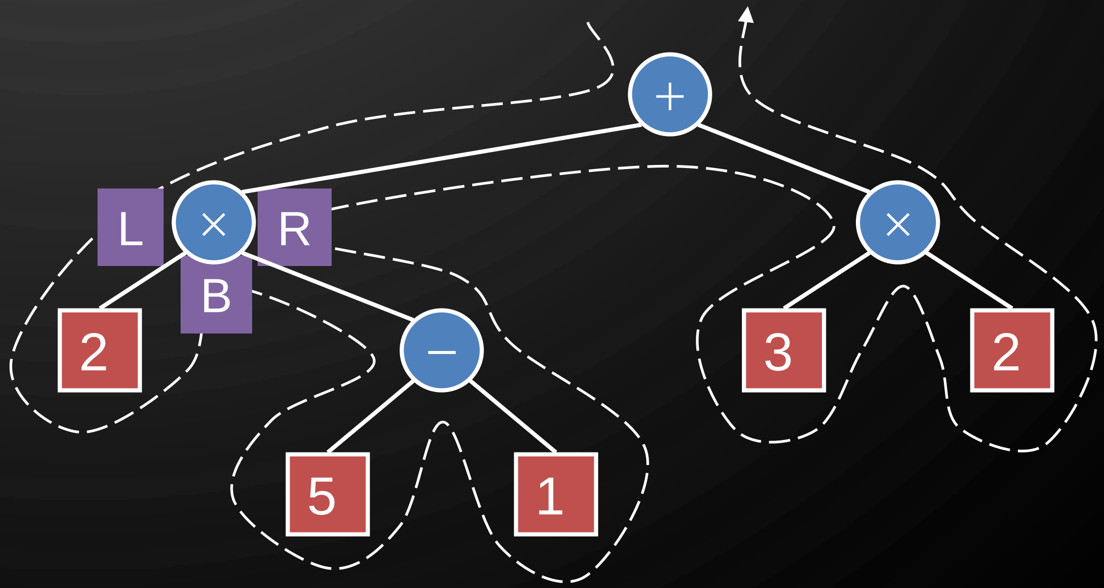
EXERCISE
ARITHMETIC EXPRESSIONS

- Draw an expression tree that has
  - Four leaves, storing the values 1, 5, 6, and 7
  - 3 internal nodes, storing operations +, -, *, /

    operators can be used more than once, but each internal node stores only one
  - The value of the root is 21

EULER TOUR TRAVERSAL

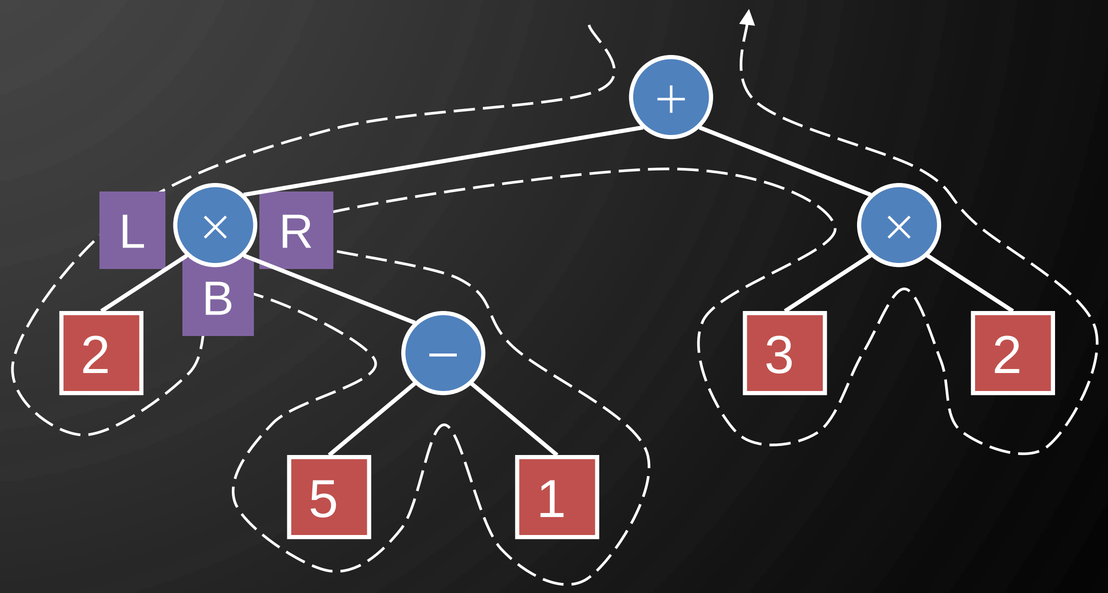
- Generic traversal of a binary tree
- Includes as special cases the preorder, postorder and inorder traversals
- Walk around the tree and visit each node three times:
  - on the left (preorder)
  - from below (inorder)
  - on the right (postorder)



EULER TOUR TRAVERSAL

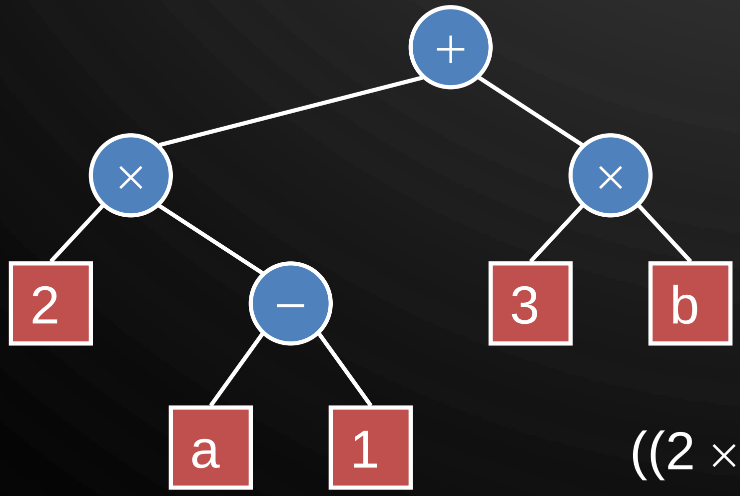
```
Algorithm eulerTour
Input: Tree T
1.eulerTour(T, T.root())
```

```
Algorithm eulerTour
Input: Tree T, Position p
1. left-visit-action(p)
2. if T.left(p)≠null then
3.     eulerTour(T, T.left(p))
4. bottom-visit-action(p)
5. if T.right(p)≠null then
6.     eulerTour(T, T.right(p))
7. right-visit-action(p)
```



APPLICATION
PRINT ARITHMETIC EXPRESSIONS

- Specialization of an Euler Tour traversal
  - Left-visit: if node is internal, print "("
  - Bottom-visit: print value or operator stored at node
  - Right-visit: if node is internal, print ")"



$((2 \times (a - 1)) + (3 \times b))$

INTERVIEW QUESTION 1

- Implement a function to check if a binary tree is balanced. For the purposes of this question, a balanced tree is defined to be a tree such that the heights of the two subtrees of any node never differ by more than one.

GAYLE LAAKMANN MCDOWELL, "CRACKING THE CODE INTERVIEW: 150 PROGRAMMING QUESTIONS AND SOLUTIONS", 5TH EDITION, CAREERCUP PUBLISHING, 2011.

INTERVIEW QUESTION 2

- Given a binary tree, design an algorithm which creates a linked list of all the nodes at each depth (e.g., if you have a tree with depth D, you'll have D linked lists).

GAYLE LAAKMANN MCDOWELL, "CRACKING THE CODE INTERVIEW: 150 PROGRAMMING QUESTIONS AND SOLUTIONS", 5TH EDITION, CAREERCUP PUBLISHING, 2011.