CHAPTER 18 RECURSION

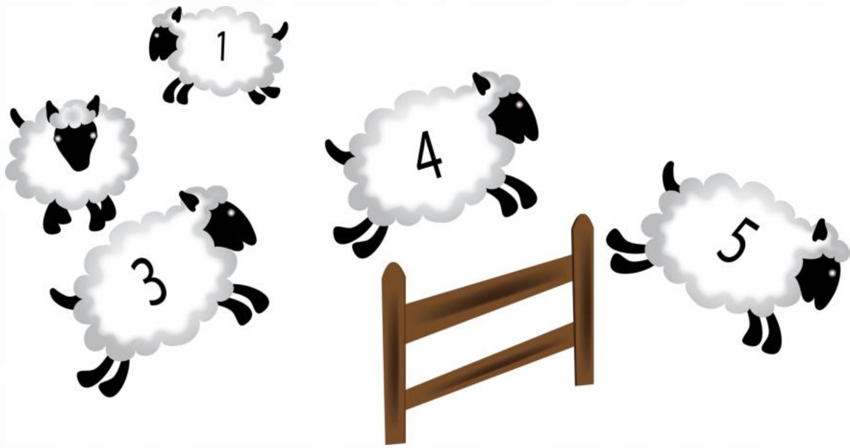
ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH INTRODUCTION TO JAVA PROGRAMMING, LIANG (PEARSON 2014)

OVERVIEW

- **Recursion** is an algorithmic technique where a function calls itself directly or indirectly.
- Why learn recursion?
  - New mode of thinking.
  - Powerful programming paradigm.
- Many computations are naturally self-referential.
  - A folder containing files and **other folders**.
  - Mathematical sequences - $f_{n+1} = \boldsymbol{f_n} + 1$ (whole numbers)
  - Exploring mazes – make a step in the maze, then keep **exploring the maze**

EXAMPLE
COUNTING

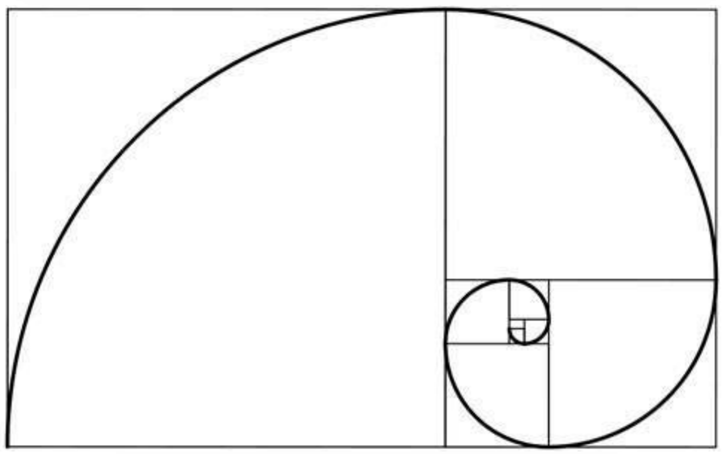
- Lets take an example of counting from 0
- The next number is the previous number plus 1, or mathematically: $f_n = f_{n-1} + 1$, where $f_0 = 0$
- So lets compute f_5
  - $f_5 = f_4 + 1$, this would be great if we knew f_4, so lets expand it
  - $f_5 = (f_3 + 1) + 1$, then
  - $f_5 = \left((f_2 + 1) + 1\right) + 1$, then
  - $f_5 = \left(\left((f_1 + 1) + 1\right) + 1\right) + 1$, then
  - $f_5 = \left(\left(\left((f_0 + 1) + 1\right) + 1\right) + 1\right) + 1$, then finally
  - $f_5 = \left(\left(\left(\left((0) + 1\right) + 1\right) + 1\right) + 1\right) + 1 = 5$



PRACTICE RECURSIVE FORMULAS
FIBONACCI SEQUENCE

- The Fibonacci Sequence is used in various places in mathematics and computer science

  $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0, f_1 = 1$
- Expand and evaluate f_6, work with a partner and show your work



HOW DO WE DO RECURSION IN CODE?

- Simply call the function within its own body

```
public static void foo() {
   //possibly do some stuff
   foo(); //This example of recursion
   //prossibly do some more stuff
}
```

EXAMPLE
COUNTING

```
//This function counts using recursion
public static int recursiveCount(int n) {
   //f_0 = 0
   if(n == 0)
     return 0;
   //f_n = f_{n-1} + 1
   return recursiveCount(n-1) + 1;
}
```

- Together lets trace
  System.out.println(recursiveCount(3));
- recursiveCount(3)
  - return recursiveCount(2) + 1
    - return recursiveCount(1) + 1;
      - return recursiveCount(0) + 1;
        - return 0;
      - return 0 + 1;
    - return 1 + 1;
  - return 2 + 1;
- 3

PRACTICE RECURSIVE CODE
FIBONACCI SEQUENCE

- Write a Java function Fibonacci for

$$f_n = f_{n-1} + f_{n-2}$$

```
public static int Fibonacci(int n) {
  if(n == 0) return 0;
  if(n == 1) return 1;
  return Fibonacci(n-1) + Fibonacci(n-2);
}
```

- Practice tracing Fibonacci(3)
- Fibonacci(3)
  - return Fibonacci(2) + Fibonacci(1);
    - return Fibonacci(1) + Fibonacci(0);
      - return 1;
    - return 1 + Fibonacci(0);
      - return 0;
    - return 1 + 0;
  - return 1 + Fibonacci(1);
    - return 1;
  - return 1 + 1;
- 2

CHARACTERISTICS OF RECURSION

- All recursive methods have the following characteristics:
  - One or more **base cases** (the simplest case) are used to stop recursion.
  - Every recursive call reduces the original problem, bringing it increasingly closer to a base case until it becomes that case.
- In general, to solve a problem using recursion, you break it into **subproblems**. If a subproblem resembles the original problem, you can apply the same approach to solve the subproblem recursively. This subproblem is almost the same as the original problem in nature with a smaller size.

DESIGNING RECURSIVE FUNCTIONS

- Identify the base case
  - The base case is the part of the recursion not defined in terms of itself, e.g., $f_0 = 0, f_1 = 1$
  - This is when the recursion stops! If you forget your base case, then the world will end
    - Really an infinite series of function calls until your computer crashes (if it ever does)
- Identify the recursive process
  - This is the algorithmic process or algorithmic formula
- Write the code

PRACTICE DESIGNING RECURSIVE FUNCTIONS
GREATEST COMMON DENOMINATOR (GCD)

- GCD
  - For two integers p and q, if q divides p then the GCD of p and q is q
  - Otherwise the GCD of p and q is the same as q and $p\%q$
- Step 1: Identify the base case
  - $q = 0$ implies that the GCD is p
- Step 2: Identify the recursive step
  - $GCD(q, p\%q)$
- Step 3: Code

```
public static int gcd(
        int p, int q) {
  if(q == 0) return p;
  return gcd(q, p%q);
}
```

RECURSIVE GCD DEMO

```java
public class Euclid {
   public static int gcd(int p, int q) {
      if (q == 0) return p;
      else return gcd(q, p % q);
   }

   public static void main(String[] args) {
      System.out.println(gcd(1272, 216));
   }
}
```

p = 1272, q = 216

Memory gcd call - 1

gcd(1272, 216)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
}
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p = 1272, q = 216

Memory gcd call - 1

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gcd(1272, 216)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
```

$1272 = 216 \times 5 + 192$

p = 216, q = 192

Memory gcd call - 2

gcd(216, 192)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
}
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p = 1272, q = 216

Memory gcd call - 1

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```

p = 216, q = 192

Memory gcd call - 2

gcd(216, 192)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
```

p = 192, q = 24

Memory gcd call - 3

gcd(192, 24)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
}
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p = 1272, q = 216

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p = 216, q = 192

Memory gcd call - 2

gcd(216, 192)

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static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
```

p = 192, q = 24

Memory gcd call - 3

gcd(192, 24)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
```

p = 24, q = 0

Memory gcd call - 4

gcd(24, 0)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

p = 1272, q = 216

Memory gcd call - 1

gcd(1272, 216)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
```

p = 216, q = 192

Memory gcd call - 2

gcd(216, 192)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
```

p = 192, q = 24

Memory gcd call - 3

gcd(192, 24)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
```

p = 24, q = 0

Memory gcd call - 4

gcd(24, 0)

```
static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

p = 1272, q = 216

Memory gcd call - 1

gcd(1272, 216)

```
static int gcd(int p, int q) {
    if (q == 0) return p;
    else return gcd(q, p % q);
```

p = 216, q = 192

Memory gcd call - 2

gcd(216, 192)

```
static int gcd(int p, int q) {
    if (q == 0) return p;
    else return gcd(q, p % q);
```

p = 192, q = 24

Memory gcd call - 3

gcd(192, 24)

```
static int gcd(int p, int q) {
    if (q == 0) return p;
    else return gcd(q, p % q);
```

24

p = 24, q = 0

Memory gcd call - 4

gcd(24, 0)

```
static int gcd(int p, int q) {
    if (q == 0) return p;
    else return gcd(q, p % q);
}
```

p = 1272, q = 216

Memory gcd call - 1

gcd(1272, 216)

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static int gcd(int p, int q) {
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Memory gcd call - 2

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24

```
public class Euclid {
  public static int gcd(int p, int q) {
    if (q == 0) return p;
    else return gcd(q, p % q);
  }

  public static void main(String[] args) {
    System.out.println(gcd(p, q));
  }
}
```

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RECURSIVE HELPER METHODS

- Many of the problems presented in the early chapters can be solved using recursion if you think recursively. For example, the palindrome problem can be solved recursively as follows:

```
public static boolean isPalindrome(String s) {
  if (s.length() <= 1) // Base case
    return true;
  else if (s.charAt(0) != s.charAt(s.length() - 1)) // Base case
    return false;
  else
    return isPalindrome(s.substring(1, s.length() - 1));
}
```

RECURSIVE HELPER METHODS

- The preceding recursive `isPalindrome` method is not efficient, because it creates a new string for every recursive call. To avoid creating new strings, use a helper method:

```
public static boolean isPalindrome(String s) {
  return isPalindrome(s, 0, s.length() - 1);
}
public static boolean isPalindrome(String s, int low, int high) {
  if (high <= low) // Base case
    return true;
  else if (s.charAt(low) != s.charAt(high)) // Base case
    return false;
  else
    return isPalindrome(s, low + 1, high - 1);
}
```

EXERCISE – PROGRAM TOGETHER

DOWNLOAD STDDRAW – LETS DRAW A TREE!

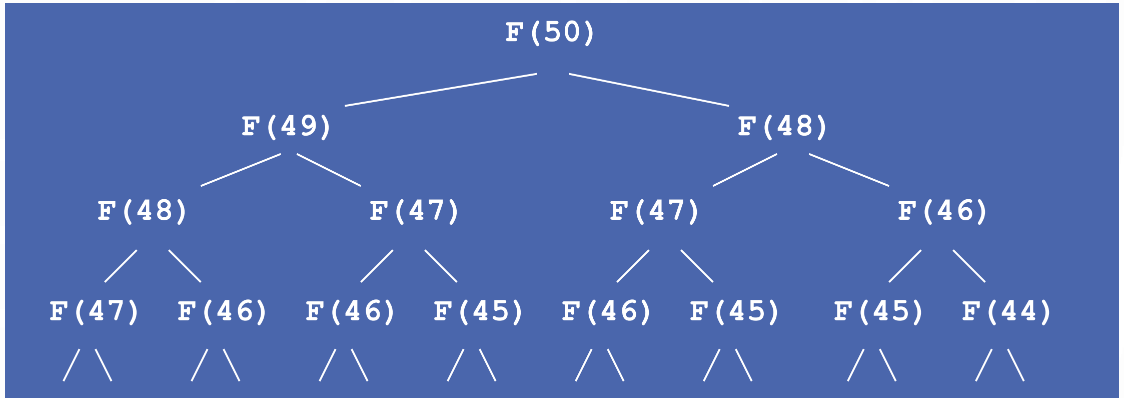
CODE

```java
import java.util.Scanner;
public class Tree {
  public static void drawTree(int o) {
    drawTree(o, 0., 0., 60,
      Math.PI/2, 35*Math.PI/180);
  }
  public static void drawTree(
      int o, double x,
      double y, double l,
      double t1, double t2) {
    if(o < 0) return;
    double x2 = x + l * Math.cos(t1);
    double y2 = y + l * Math.sin(t1);
    StdDraw.line(x, y, x2, y2);
    drawTree(o - 1, x2, y2, l*0.6, t1 + t2, t2);
    drawTree(o - 1, x2, y2, l*0.6, t1 - t2, t2);
  }
  public static void main(String args[]) {
    Scanner in = new Scanner(System.in);
    System.out.print("Enter an order: ");
    int order = in.nextInt();
    StdDraw.setXscale(-200, 200);
    StdDraw.setYscale(0, 400);
    StdDraw.enableDoubleBuffering();
    drawTree(order);
    StdDraw.show();
  }
}
```

DOWNSIDE OF RECURSION

- Recursion is not always efficient!
- Take for instance, the Fibonacci sequence

```java
public static long F(int n) {
    if (n == 0) return 0;
    if (n == 1) return 1;
    return F(n-1) + F(n-2);
}
```



recursion tree for naïve Fibonacci function

- F(50) is called once.
- F(49) is called once.
- F(48) is called 2 times.
- F(47) is called 3 times.

…

- F(1) is called 12,586,269,025 times.

BEST PRACTICE
CONVERT RECURSIVE ALGORITHMS TO ITERATIVE ONES

- Try to do this whenever possible
- Example Fibonacci Sequence

```
//This is an example conversion. You can be even more efficient!
public static long F(int n) {
  if (n == 0) return 0;
  if (n == 0) return 1;
  int fn2 = 0;
  int fn1 = 1;
  int fn = 1;
  // Iterative means repetition until failure condition,
  // typically done with loops and not recursion
  for (int i = 2; i <= n; i++) {
    fn = fn1 + fn2;
    fn2 = fn1;
    fn1 = fn;
  }
  return fn;
}
```

OR DO TAIL RECURSION

- Tail recursion is when the last operation of a function is the recursive call
- Example with factorial:

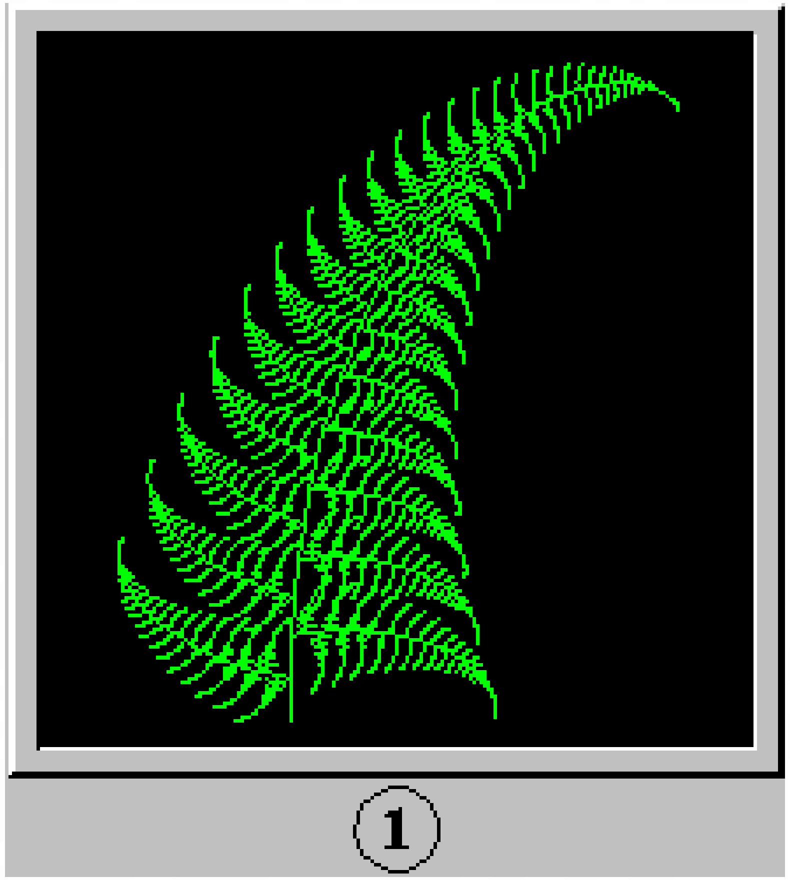
$$f_n = n * f_{n-1}$$

```
public static long factorial(
        int n) {
    if(n == 0) return 1;
    else
        return n*factorial(n-1);
}
```

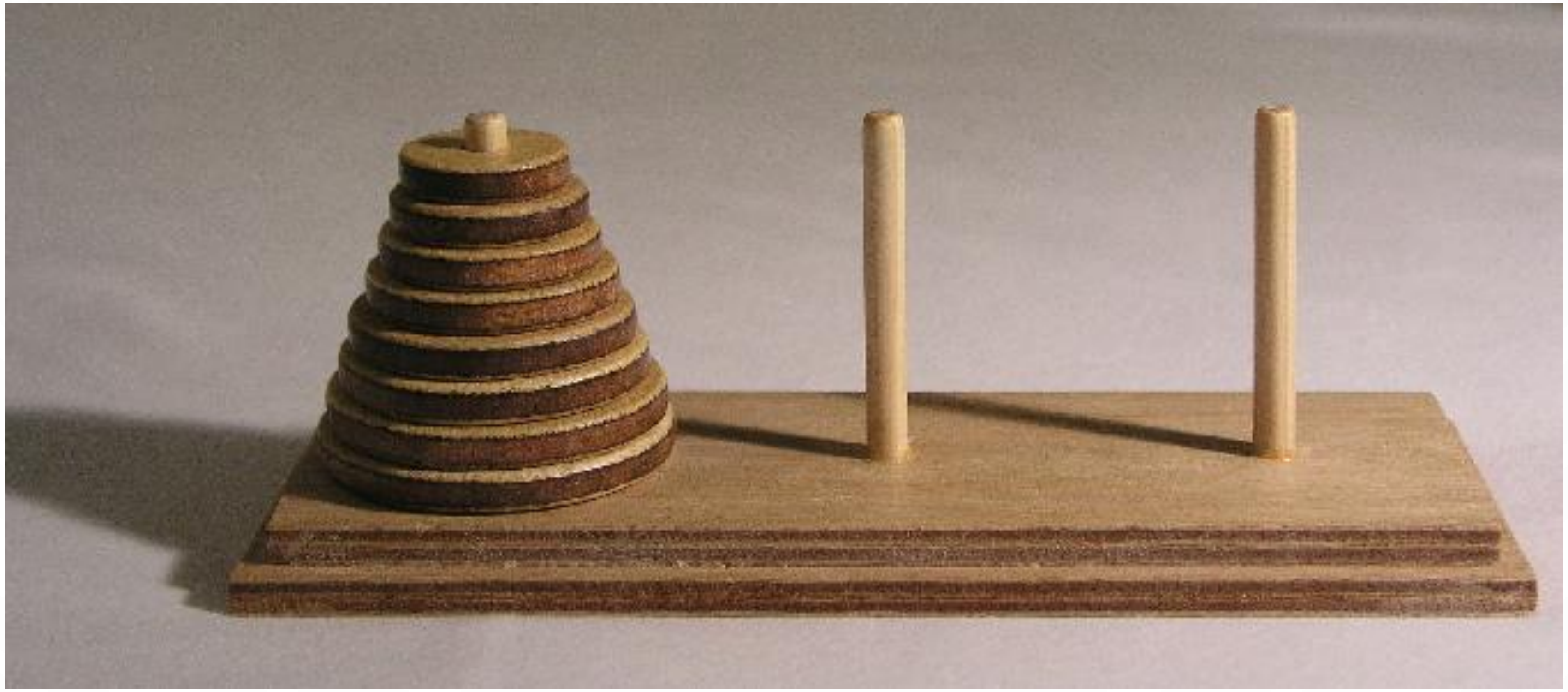
```
public static long factorial(
        int n) {
    return factorial(n, 1);
}
public static long factorial(
        int n, int result) {
    if(n == 0) return result;
    else
        return factorial(
            n - 1, n * result);
}
```

SUMMARY

- How to write simple recursive programs?
  - Base case, reduction step.
- Trace the execution of a recursive program.
- Why learn recursion?
  - New mode of thinking.
  - Powerful programming tool



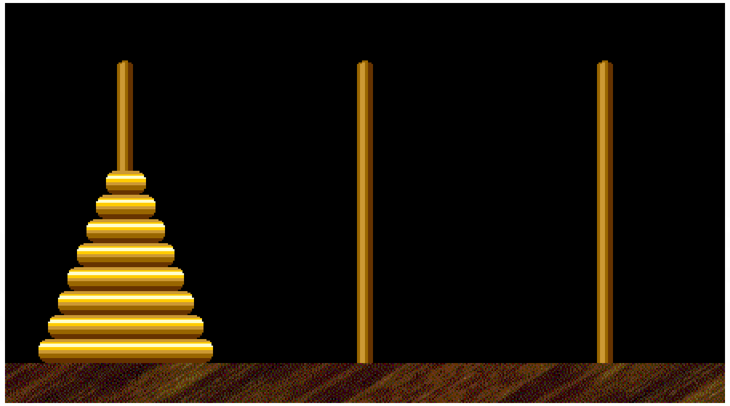
TOWERS OF HANOI



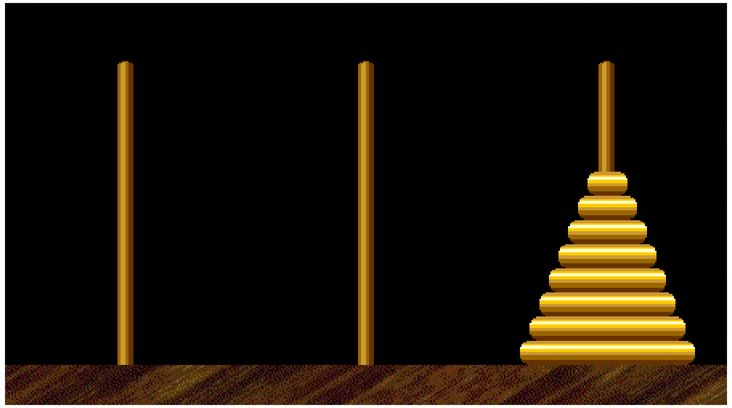
http://en.wikipedia.org/wiki/Image:Hanoiklein.jpg

PRACTICE
TOWERS OF HANOI

- Design recursive algorithm to move all the discs from the leftmost peg to the rightmost one.
  - Only one disc may be moved at a time.
  - A disc can be placed either on empty peg or on top of a larger disc.

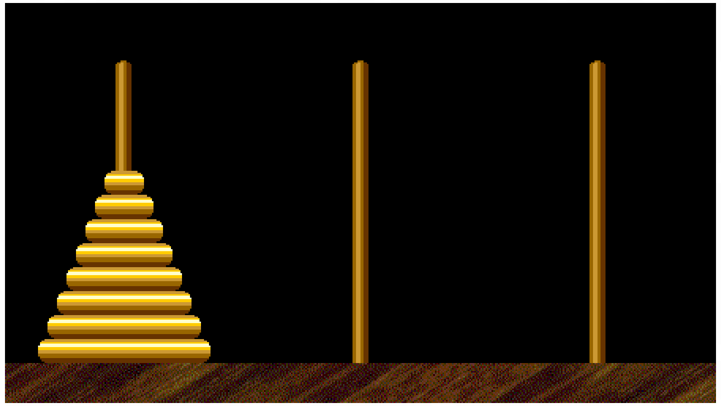


start

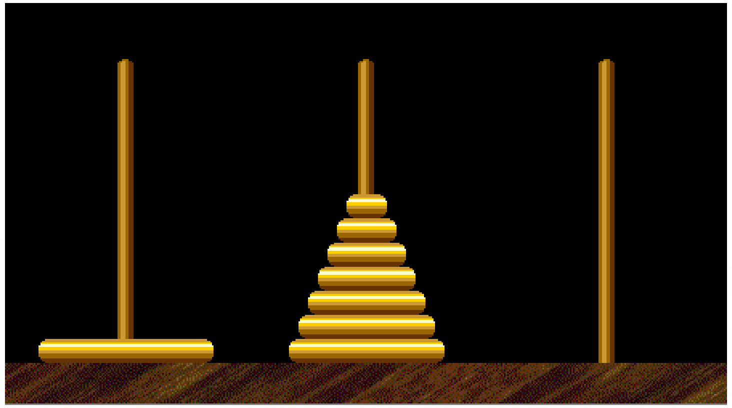


finish

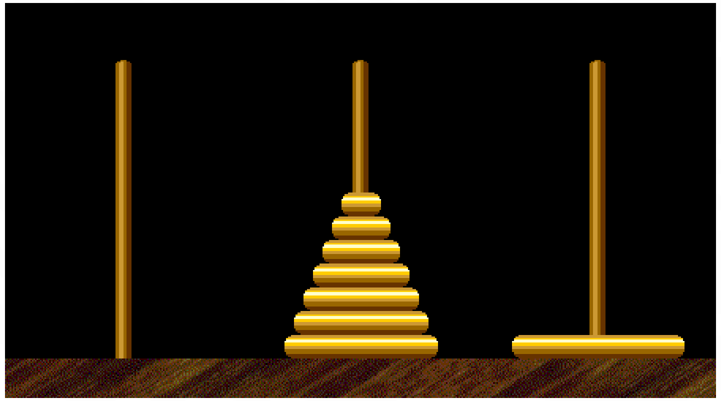
SOLUTION
TOWERS OF HANOI



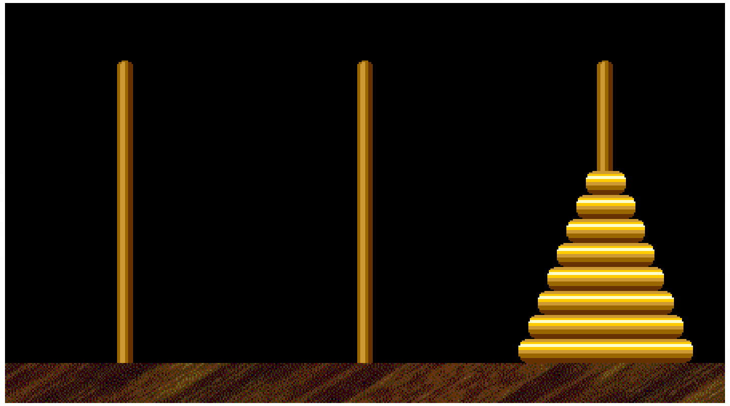
Move n-1 smallest discs right.



Move largest disc right.



Move n-1 smallest discs right.



CODE

TOWERS OF HANOI

```
public class TowersOfHanoi {
  public static void moves(int n,
      char from, char to, char aux) {
    if (n == 0) return;
    moves(n-1, from, aux, to);
    System.out.printf("Move disk %d from peg %c to peg %c", n, from, to);
    moves(n-1, aux, to, from);
  }

  public static void main(String[] args) {
    moves(3, 'A', 'C', 'B');
  }
}
```

RECURSION TREE (FUNCTION TRACE)

TOWERS OF HANOI