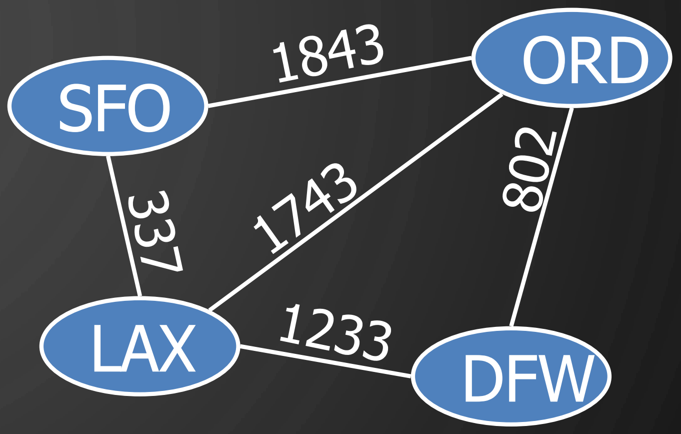
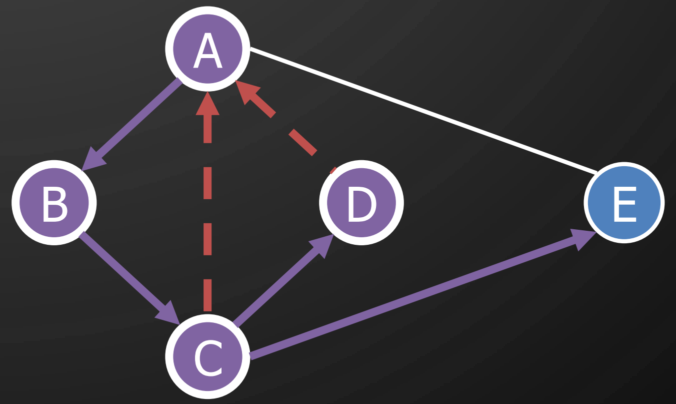
CHAPTER 14
GRAPH ALGORITHMS

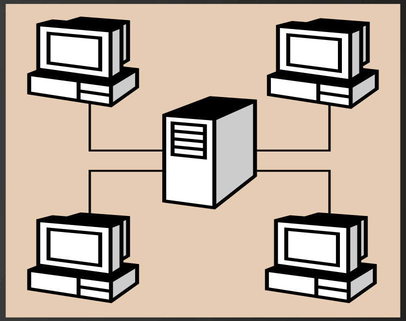


ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN JAVA, GOODRICH, TAMASSIA AND GOLDWASSER (WILEY 2016)

DEPTH-FIRST SEARCH



DEPTH-FIRST SEARCH



- **Depth-first search (DFS)** is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- DFS on a graph with n vertices and m edges takes $O(n+m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs as what Euler tour is to binary trees

DFS ALGORITHM FROM A VERTEX

Algorithm DFS(G, u)

Input: A graph G and a vertex u of G

Output: A collection of vertices reachable from u, with their discovery edges

1. Mark u as visited
2. **for each** edge $e = (u, v) \in G$.outgoingEdges(u) **do**
3.   **if** v has not been visited **then**
4.   Record e as a discovery edge for v
5.   DFS(G, v)

EXAMPLE

unexplored vertex

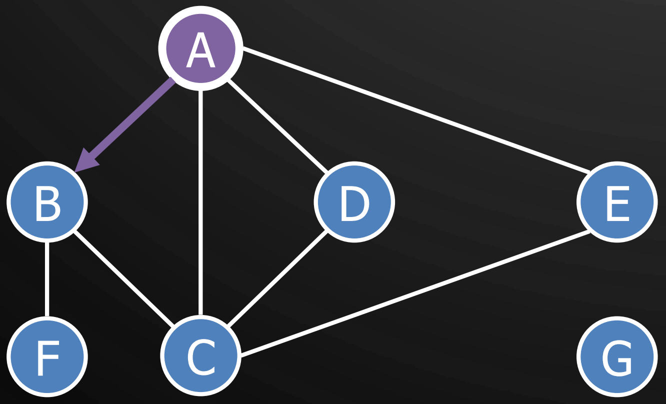
visited vertex

unexplored edge

discovery edge

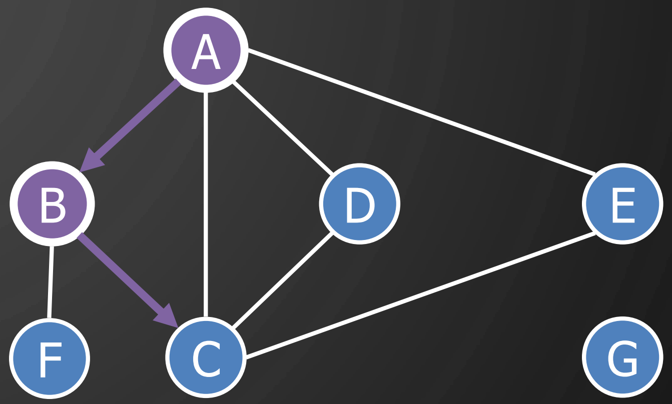
back edge

$I(A) = \{\boldsymbol{B}, C, D, E\}$

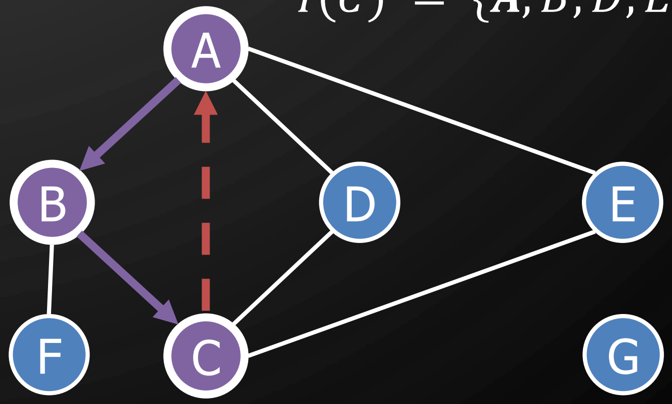


$I(B) = \{\boldsymbol{A}, C, F\}$

$I(B) = \{A, \boldsymbol{C}, F\}$



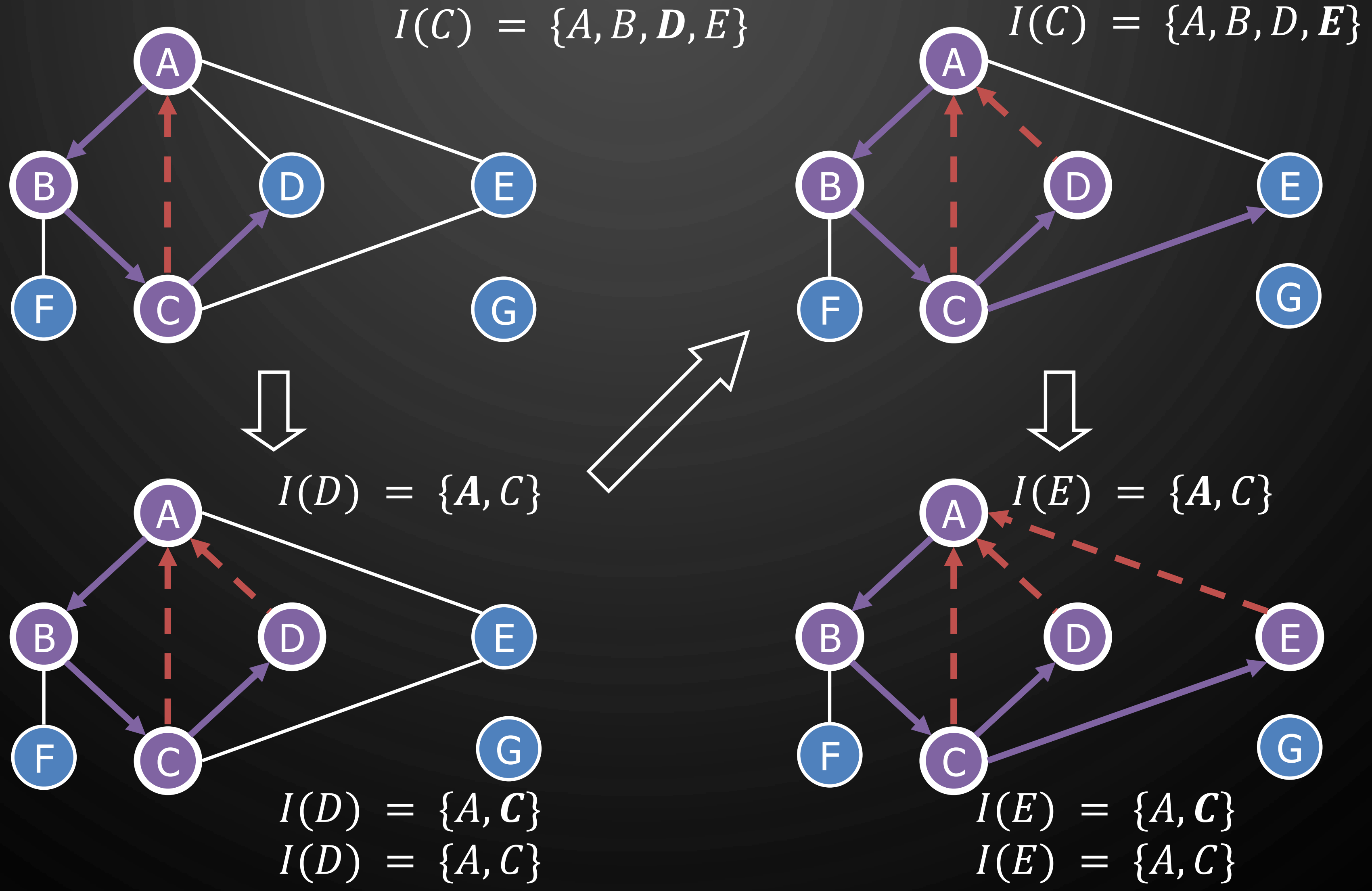
$I(C) = \{\boldsymbol{A}, B, D, E\}$



$I(C) = \{A, \boldsymbol{B}, D, E\}$

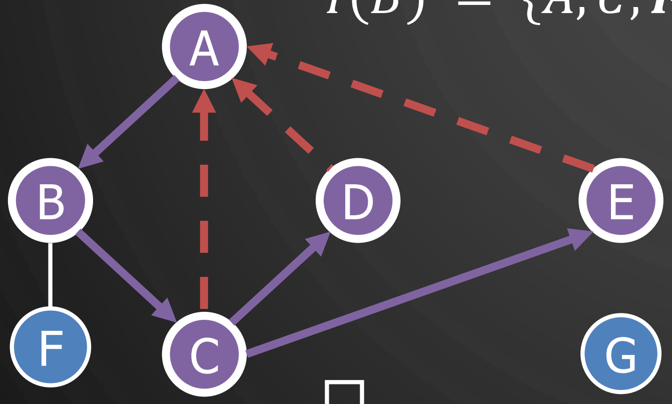
$I(C) = \{A, B, \boldsymbol{D}, E\}$

EXAMPLE

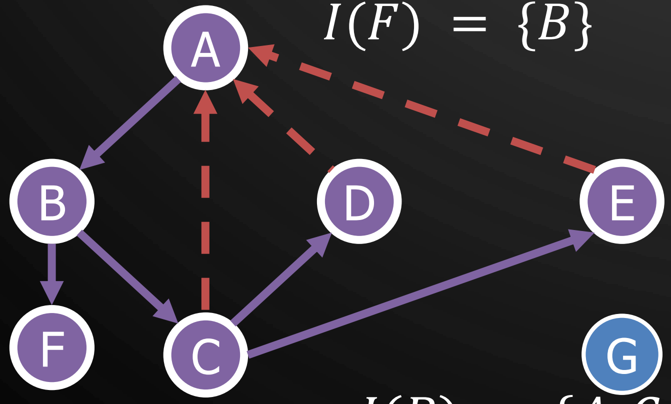


EXAMPLE

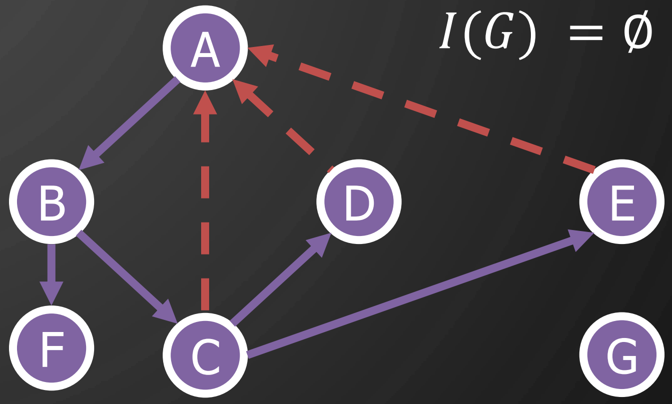
$I(C) = \{A, B, D, E\}$
$I(B) = \{A, C, \boldsymbol{F}\}$



$I(F) = \{B\}$



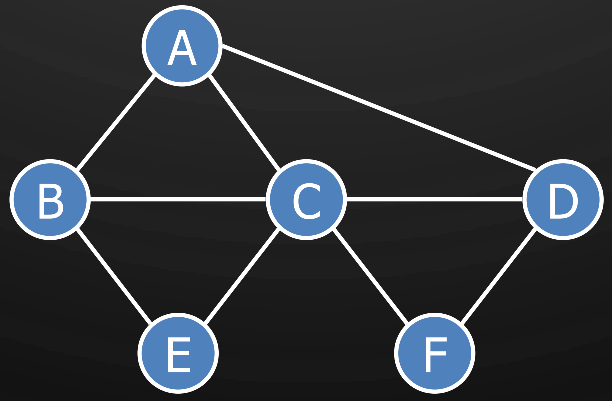
$I(B) = \{A, C, F\}$
$I(A) = \{A, B, C, D\}$



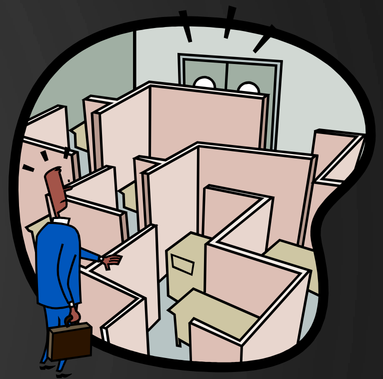
$I(G) = \emptyset$

EXERCISE
DFS ALGORITHM

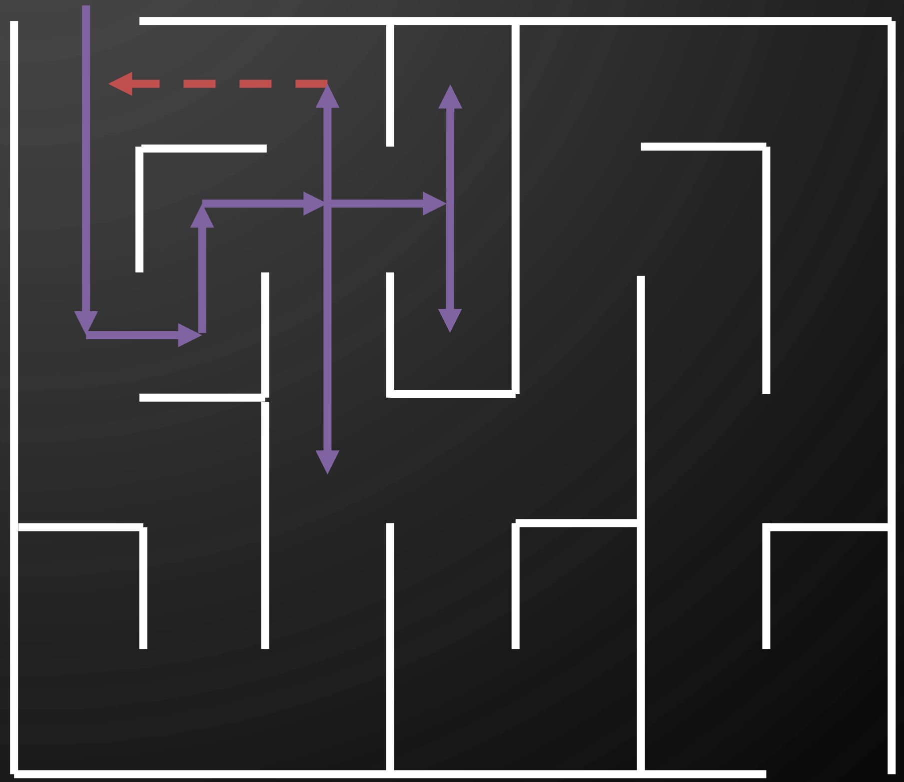
- Perform DFS of the following graph, start from vertex A
  - Assume adjacent edges are processed in alphabetical order
  - Number vertices in the order they are visited
  - Label edges as discovery or back edges



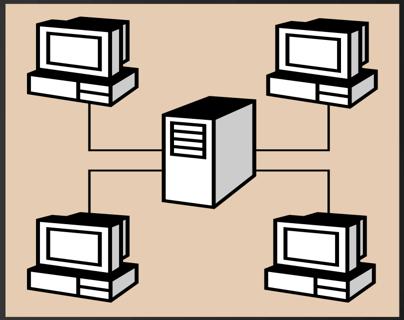
DFS AND MAZE TRAVERSAL



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



DFS ALGORITHM



- The algorithm uses a mechanism for setting and getting “labels” of vertices and edges

<u>**Algorithm** DFS(G)</u>
Input: Graph G
Output: Labeling of the edges of G as discovery edges and back edges

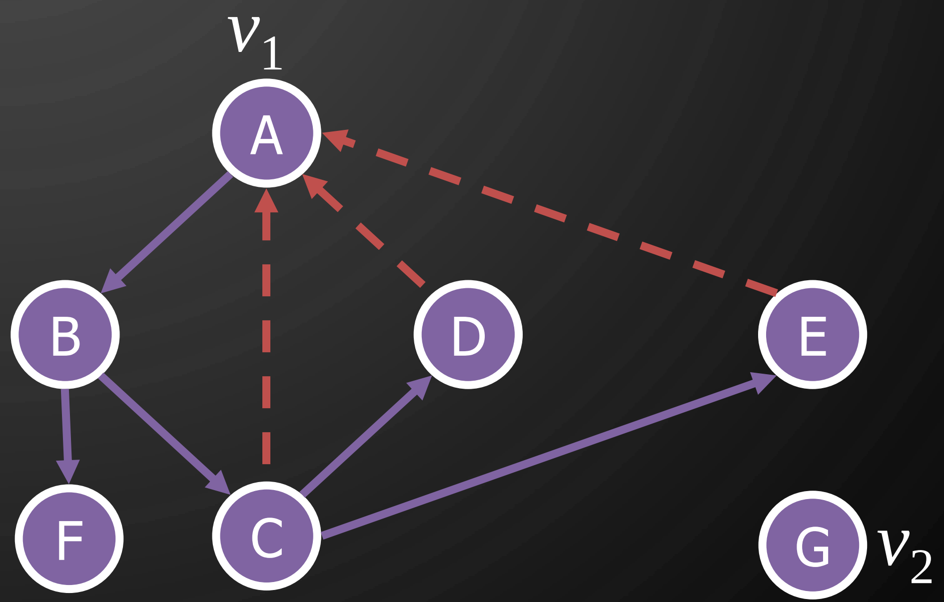
for each $v \in G$.vertices() **do**
  setLabel(v, $UNEXPLORED$)
for each $e \in G$.edges() **do**
  setLabel(e, $UNEXPLORED$)
for each $v \in G$.vertices() **do**
  if getLabel(v) $= UNEXPLORED$ **then**
    DFS(G, v)

<u>**Algorithm** DFS(G, v)</u>
Input: Graph G and a start vertex v
Output: Labeling of the edges of G in the connected component of v as discovery edges and back edges

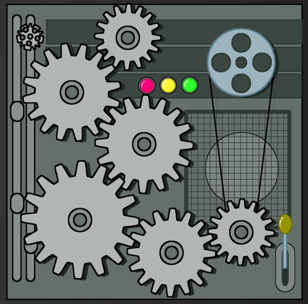
setLabel(v, $VISITED$)
for each $e \in G$.outgoingEdges(v) **do**
  if getLabel(e) $= UNEXPLORED$)
    $w \leftarrow G$.opposite(v, e)
    if getLabel(w) $= UNEXPLORED$ **then**
      setLabel(e, $DISCOVERY$)
      DFS(G, w)
    else
      setLabel(e, $BACK$)

PROPERTIES OF DFS

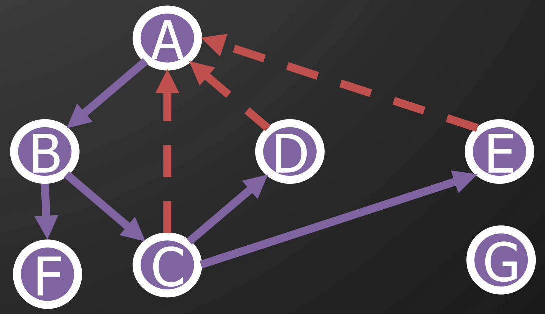
- Property 1
  - $\mathrm{DFS}(G, v)$ visits all the vertices and edges in the connected component of v
- Property 2
  - The discovery edges labeled by $\mathrm{DFS}(G, v)$ form a spanning tree of the connected component of v



ANALYSIS OF DFS



- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as $UNEXPLORED$
  - once as $VISITED$
- Each edge is labeled twice
  - once as $UNEXPLORED$
  - once as $DISCOVERY$ or $BACK$



- Function $\mathrm{DFS}(G, v)$ and the method $\mathrm{outgoingEdges}(\,)$ are called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

APPLICATION
PATH FINDING

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call $\mathrm{DFS}(G, u)$ with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G,v,z)
  setLabel(v,VISITED)
  S.push(v)
  if v = z
    return S.elements()
  for each e ∈ G.outgoingEdges(v) do
    if getLabel(e) = UNEXPLORED) then
      w ← G.opposite(v,e)
      if getLabel(w) = UNEXPLORED then
        setLabel(e,DISCOVERY)
        S.push(e)
        pathDFS(G,w)
        S.pop()
      else
        setLabel(e,BACK)
  S.pop()
```

APPLICATION

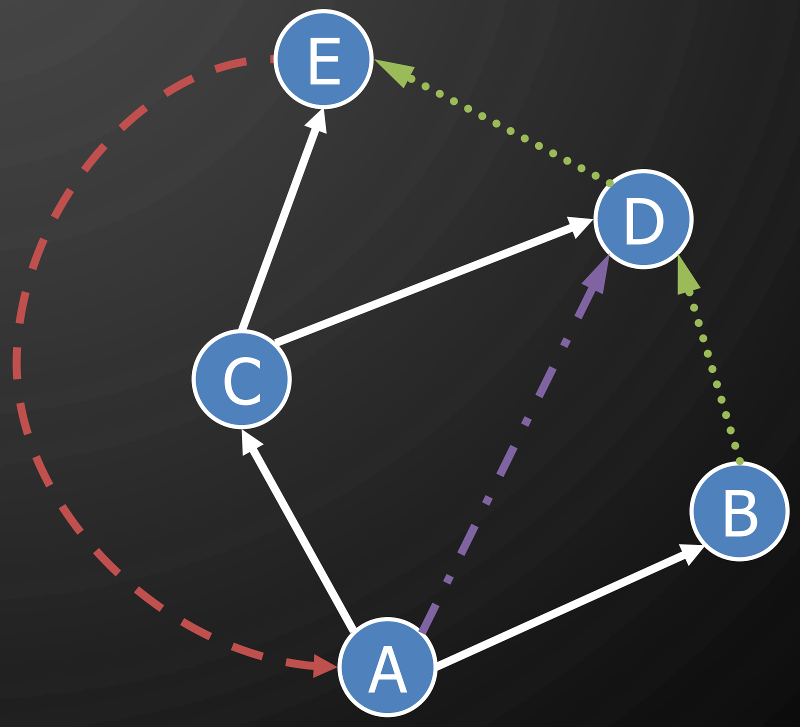
CYCLE FINDING

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex w

```
Algorithm cycleDFS(G, v)
  setLabel(v, VISITED)
  S.push(v)
  for each e ∈ G.outgoingEdges(v) do
    if getLabel(e) = UNEXPLORED) then
      w ← G.opposite(v, e)
      S.push(e)
      if getLabel(w) = UNEXPLORED then
        setLabel(e, DISCOVERY)
        cycleDFS(G, w)
        S.pop()
      else
        T ← empty stack
        repeat
          T.push(S.pop())
        until T.top() = w
        return T.elements()
  S.pop()
```

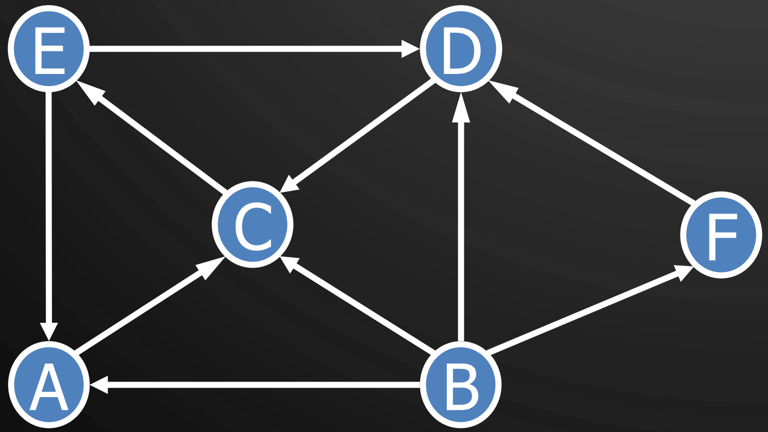
DIRECTED DFS

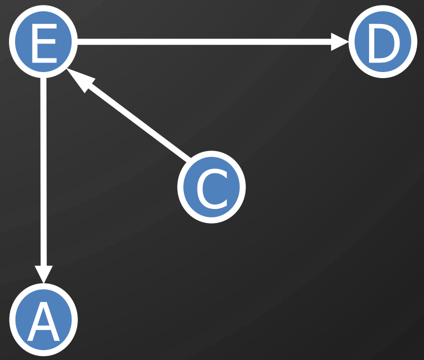
- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting at a vertex s determines the vertices reachable from s

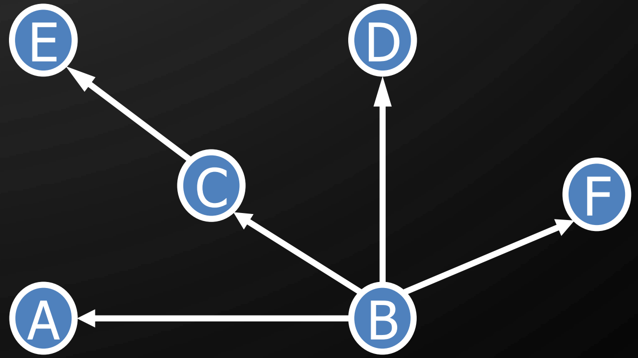


REACHABILITY

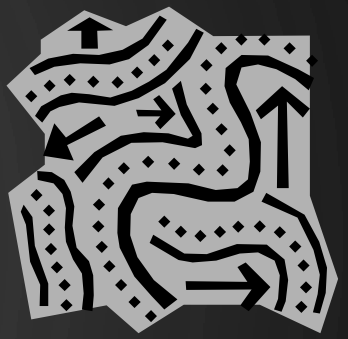
- DFS tree rooted at v: vertices reachable from v via directed paths



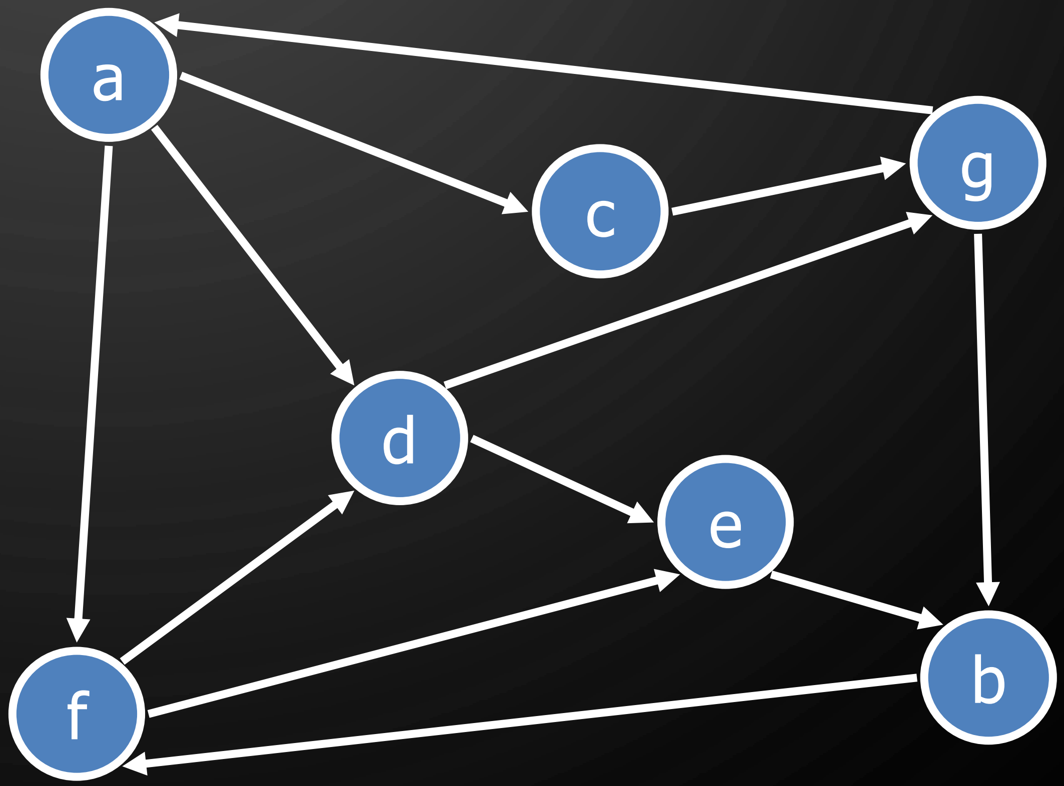




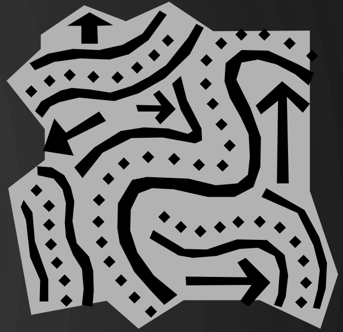
STRONG CONNECTIVITY



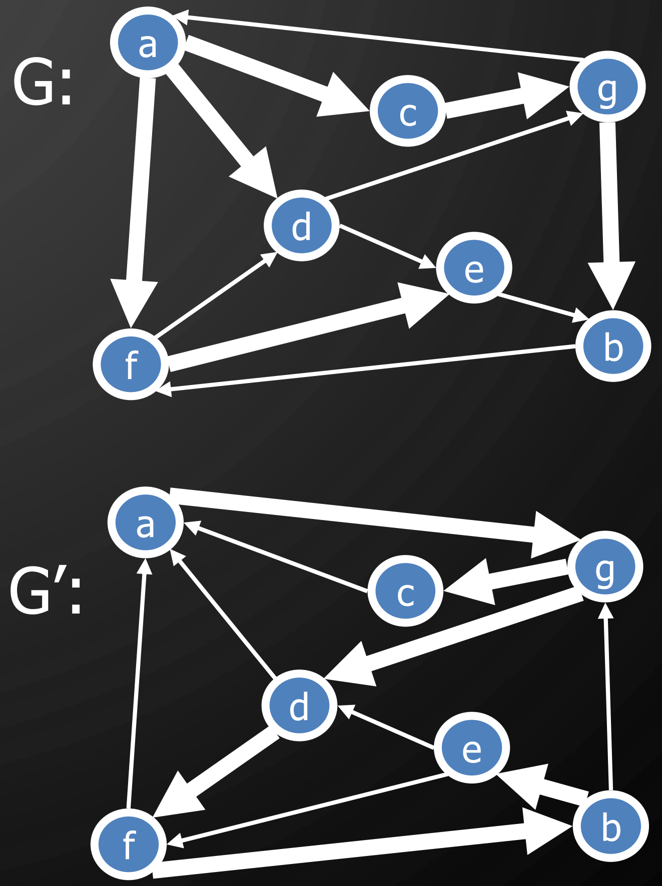
- Each vertex can reach all other vertices



STRONG CONNECTIVITY ALGORITHM

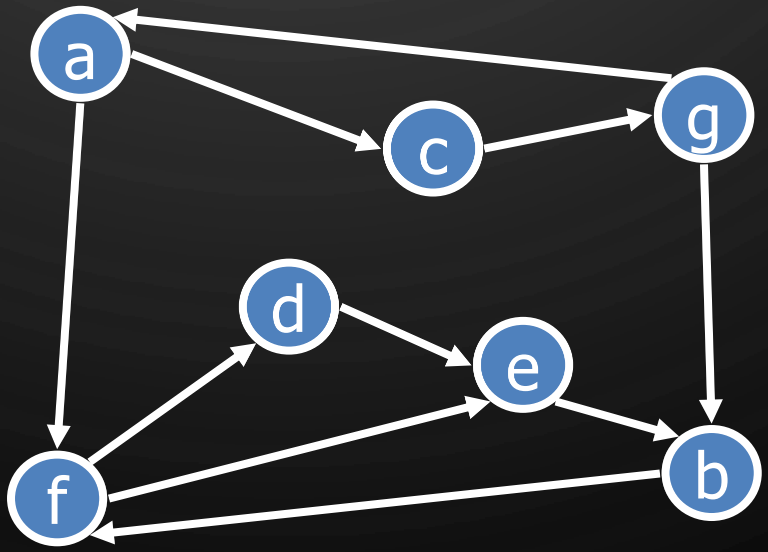


- Pick a vertex v in G
- Perform a DFS from v in G
  - If there's a w not visited, print "no"
- Let G' be G with edges reversed
- Perform a DFS from v in G'
  - If there's a w not visited, print "no"
  - Else, print "yes"
- Running time: $O(n+m)$



STRONGLY CONNECTED COMPONENTS

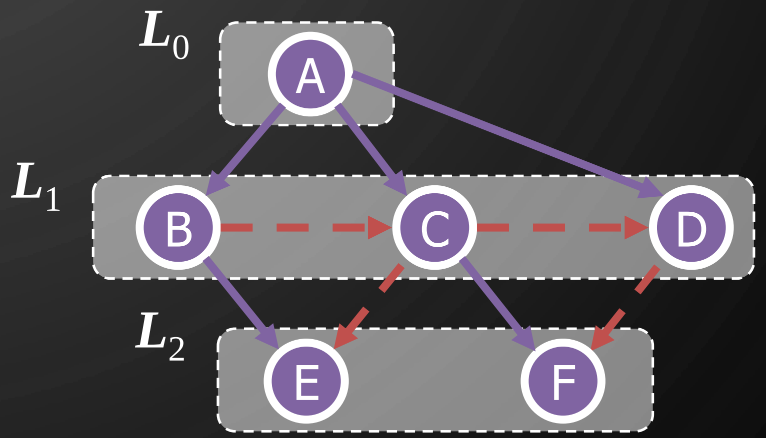
- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in $O(n+m)$ time using DFS, but is more complicated (similar to biconnectivity).



$\{a, c, g\}$

$\{f, d, e, b\}$

BREADTH-FIRST SEARCH



BREADTH-FIRST SEARCH

- **Breadth-first search (BFS)** is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with n vertices and m edges takes $O(n+m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

BFS ALGORITHM

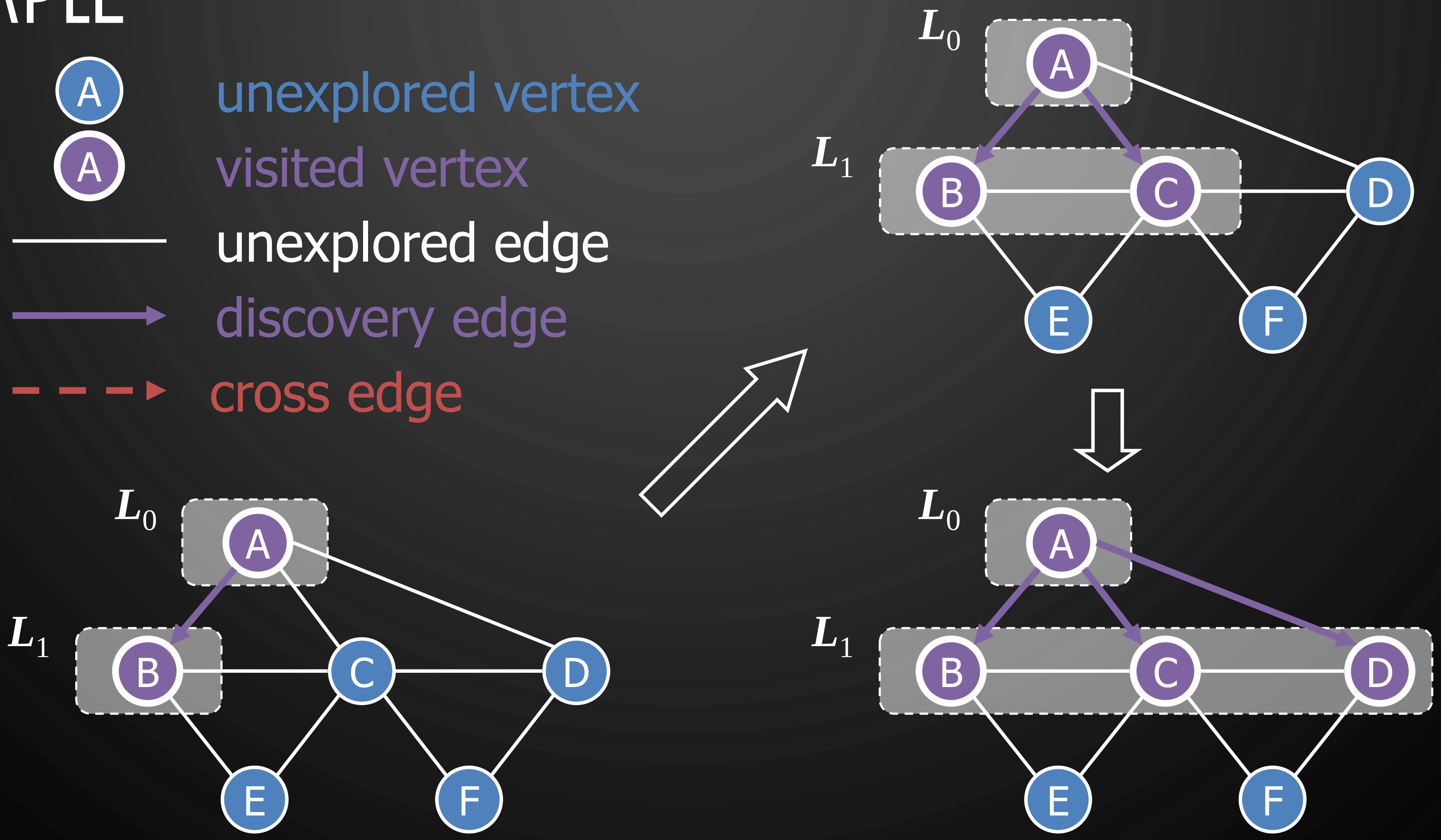
- The algorithm uses a mechanism for setting and getting “labels” of vertices and edges

```
Algorithm BFS(G)
Input: Graph G
Output: Labeling of the edges and
   partition of the vertices of G
for each v ∈ G.vertices() do
    setLabel(v, UNEXPLORED)
for each e ∈ G.edges() do
    setLabel(e, UNEXPLORED)
for each v ∈ G.vertices() do
    if getLabel(v) = UNEXPLORED then
        BFS(G, v)
```

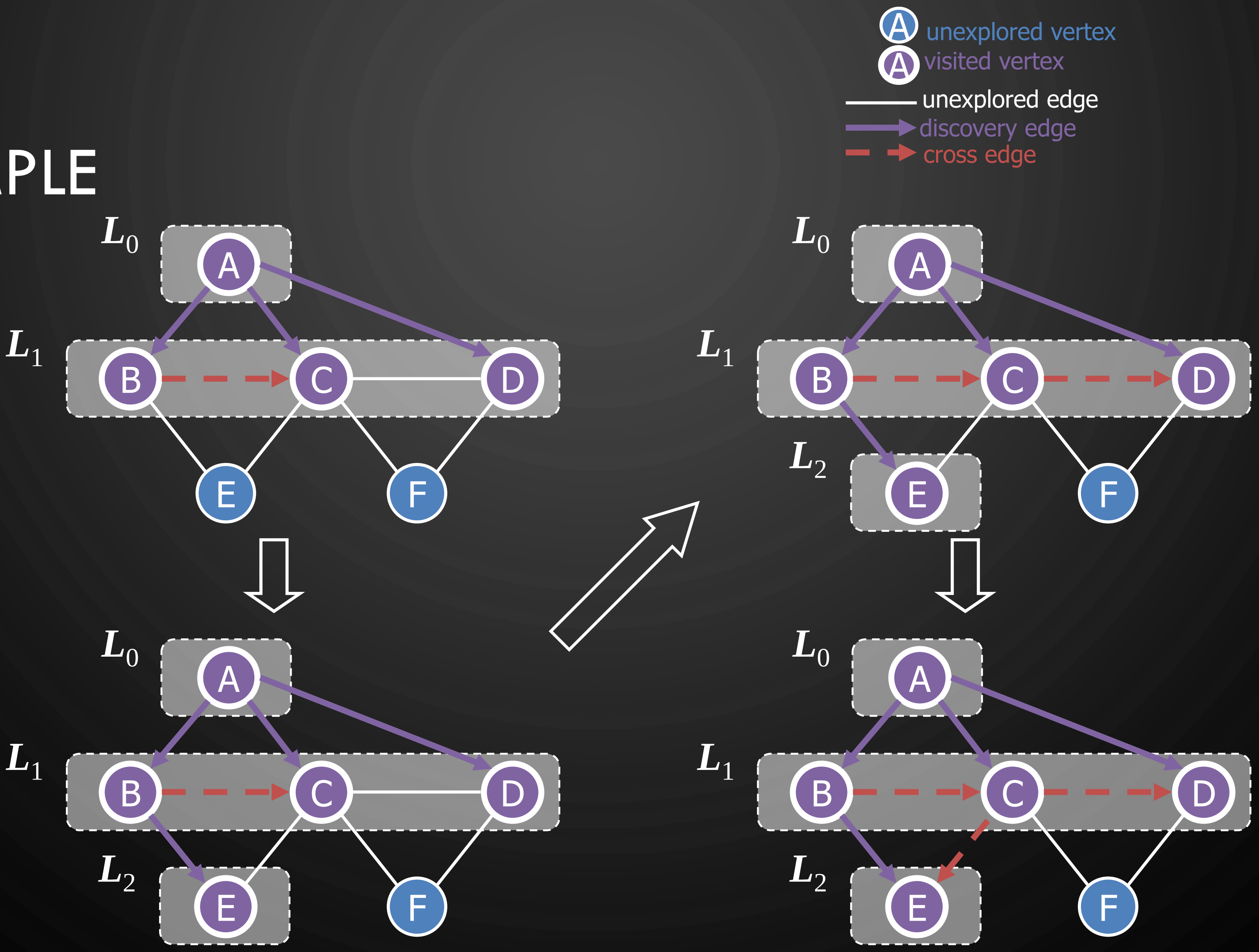
```
Algorithm BFS(G, s)
L_0 ← {s}
setLabel(s, VISITED)
i ← 0
while ¬L_i.isEmpty() do
    L_{i+1} ← ∅
    for each v ∈ L_i do
        for each e ∈ G.outgoingEdges(v) do
            if getLabel(e) = UNEXPLORED then
                w ← G.opposite(v, e)
                if getLabel(w) = UNEXPLORED then
                    setLabel(e, DISCOVERY)
                    setLabel(w, VISITED)
                    L_{i+1} ← L_{i+1} ∪ {w}
                else
                    setLabel(e, CROSS)
    i ← i + 1
```

EXAMPLE

A unexplored vertex
A visited vertex
unexplored edge
discovery edge
cross edge

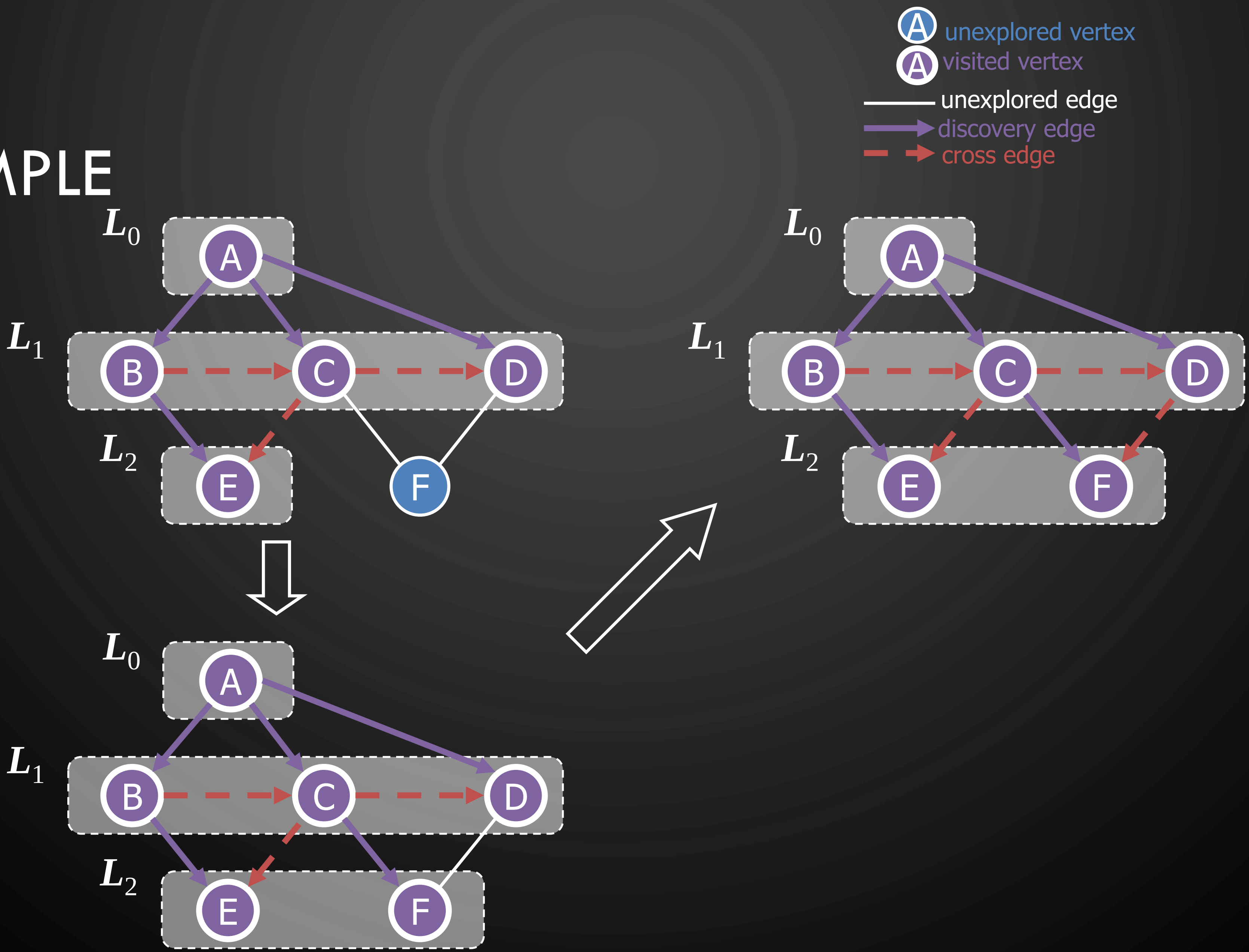


EXAMPLE



A unexplored vertex
A visited vertex
unexplored edge
discovery edge
cross edge
L_0
L_1
L_2
A
B
C
D
E
F

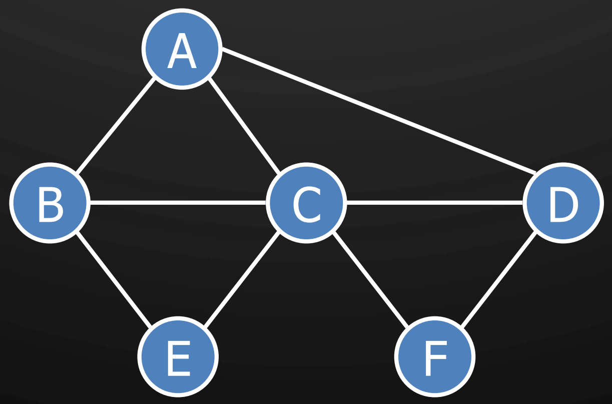
EXAMPLE



unexplored vertex
visited vertex
unexplored edge
discovery edge
cross edge
L_0
L_1
L_2
A
B
C
D
E
F

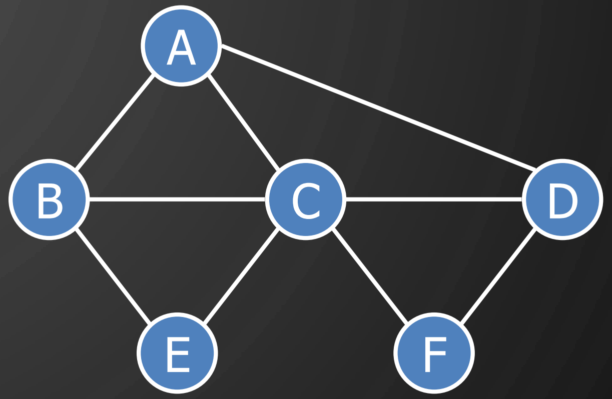
EXERCISE
BFS ALGORITHM

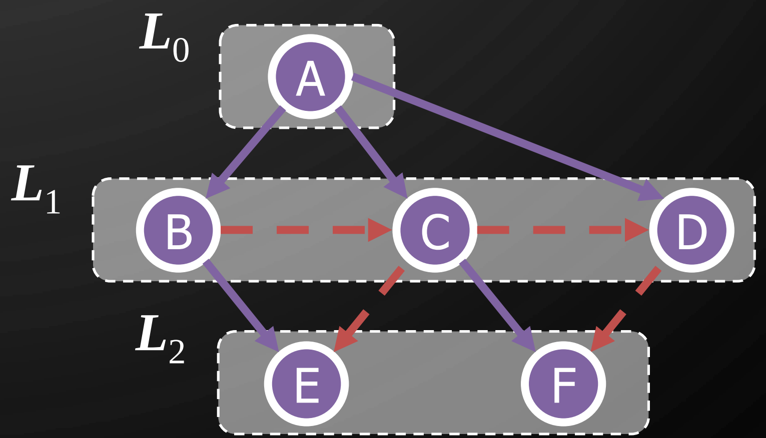
- Perform BFS of the following graph, start from vertex A
  - Assume adjacent edges are processed in alphabetical order
  - Number vertices in the order they are visited and note the level they are in
  - Label edges as discovery or cross edges



PROPERTIES

- Notation
  - G_s: connected component of s
- Property 1
  - $\mathrm{BFS}(G, s)$ visits all the vertices and edges of G_s
- Property 2
  - The discovery edges labeled by $\mathrm{BFS}(G, s)$ form a spanning tree T_s of G_s
- Property 3
  - For each vertex $v \in L_i$
    - The path of T_s from s to v has i edges
    - Every path from s to v in G_s has at least i edges





ANALYSIS

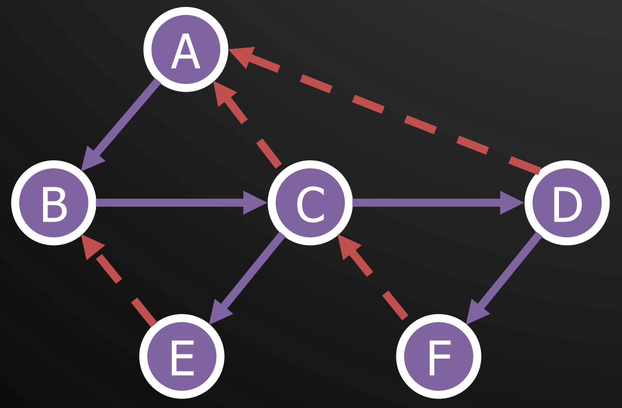
- Setting/getting a vertex/edge label takes O(1) time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method $\mathrm{outgoingEdges}(\,)$ is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

APPLICATIONS

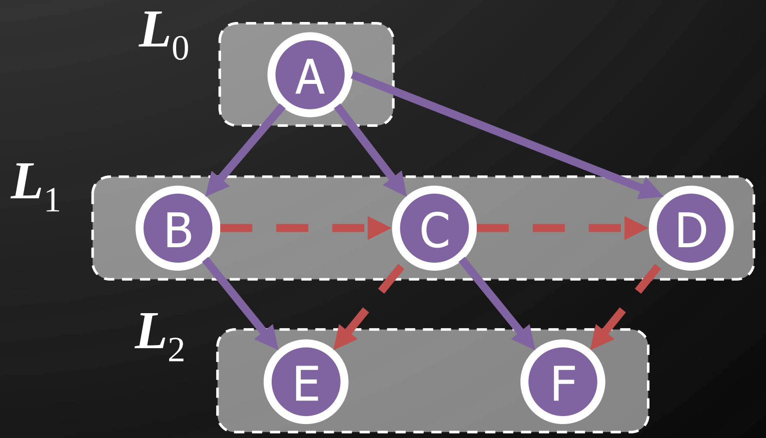
- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n + m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

DFS VS. BFS

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS



BFS

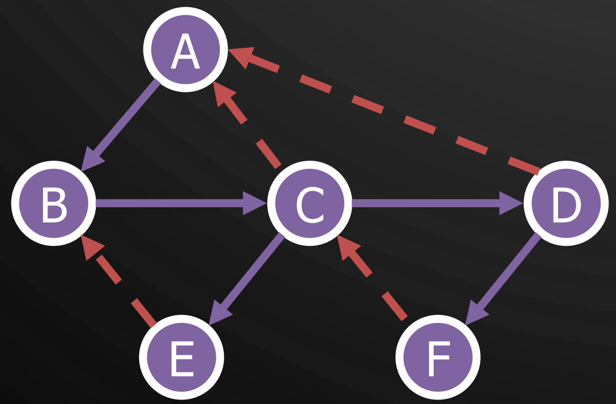
DFS VS. BFS

Back edge (v, w)

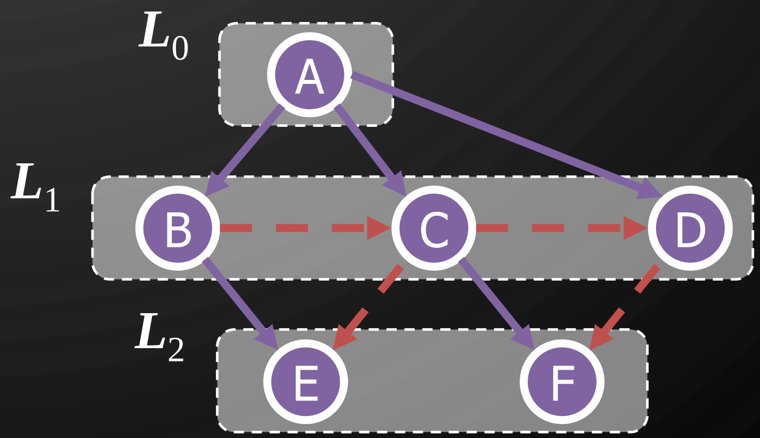
- w is an ancestor of v in the tree of discovery edges

Cross edge (v, w)

- w is in the same level as v or in the next level in the tree of discovery edges

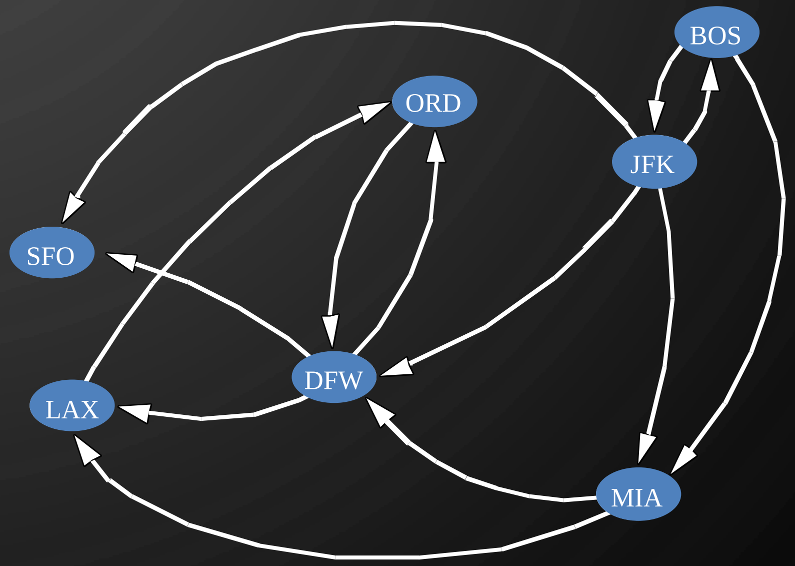


DFS



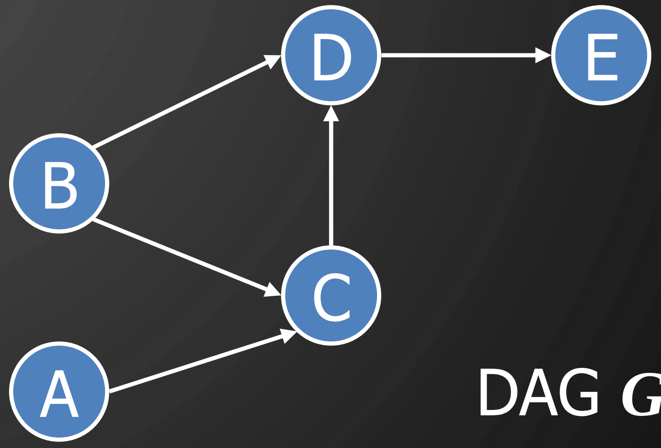
BFS

TOPOLOGICAL ORDERING

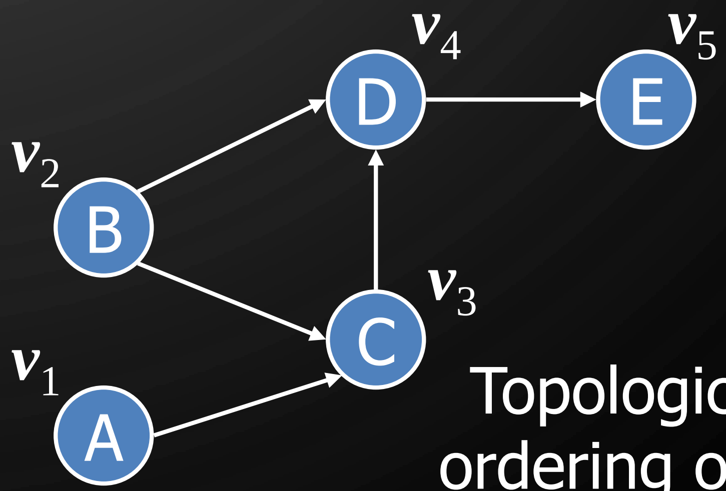


DAGS AND TOPOLOGICAL ORDERING

- A **directed acyclic graph (DAG)** is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering
  - $v_1, \dots, v_n$
  - Of the vertices such that for every edge (v_i, v_j), we have $i < j$
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints
- Theorem - A digraph admits a topological ordering if and only if it is a DAG



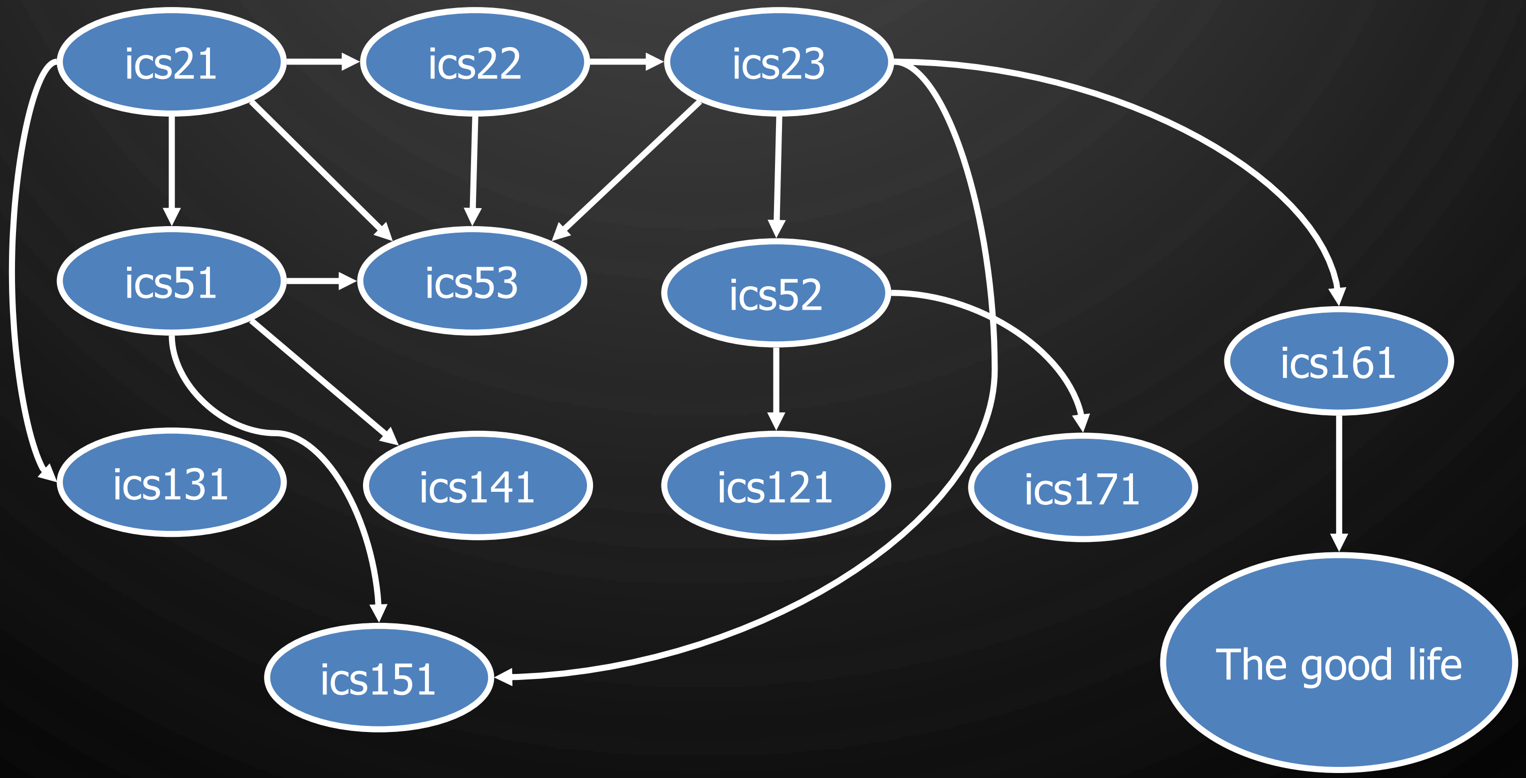
DAG G



Topological ordering of G

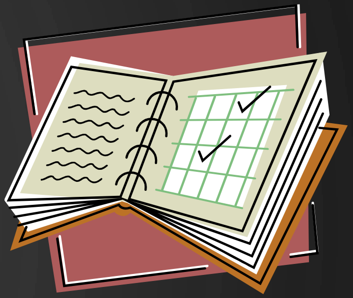
APPLICATION

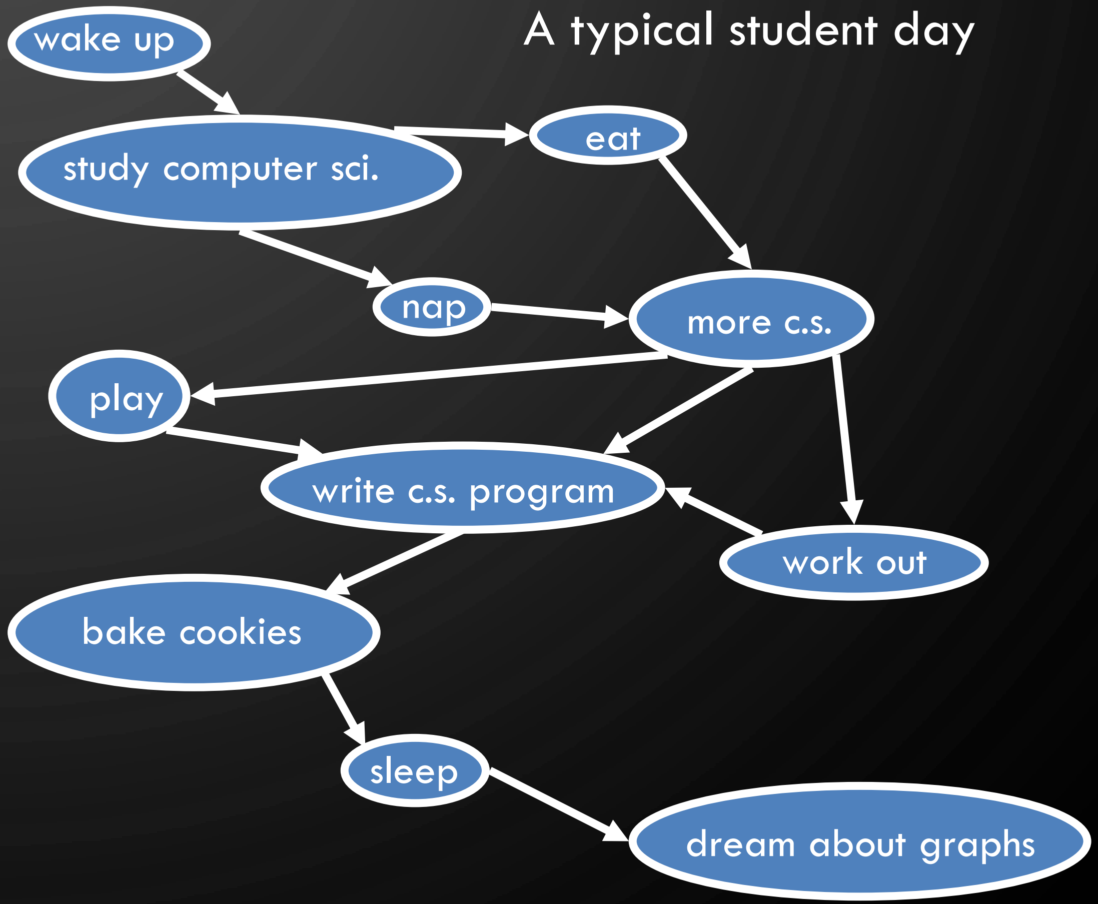
- Scheduling: edge (a, b) means task a must be completed before b can be started



EXERCISE
TOPOLOGICAL SORTING

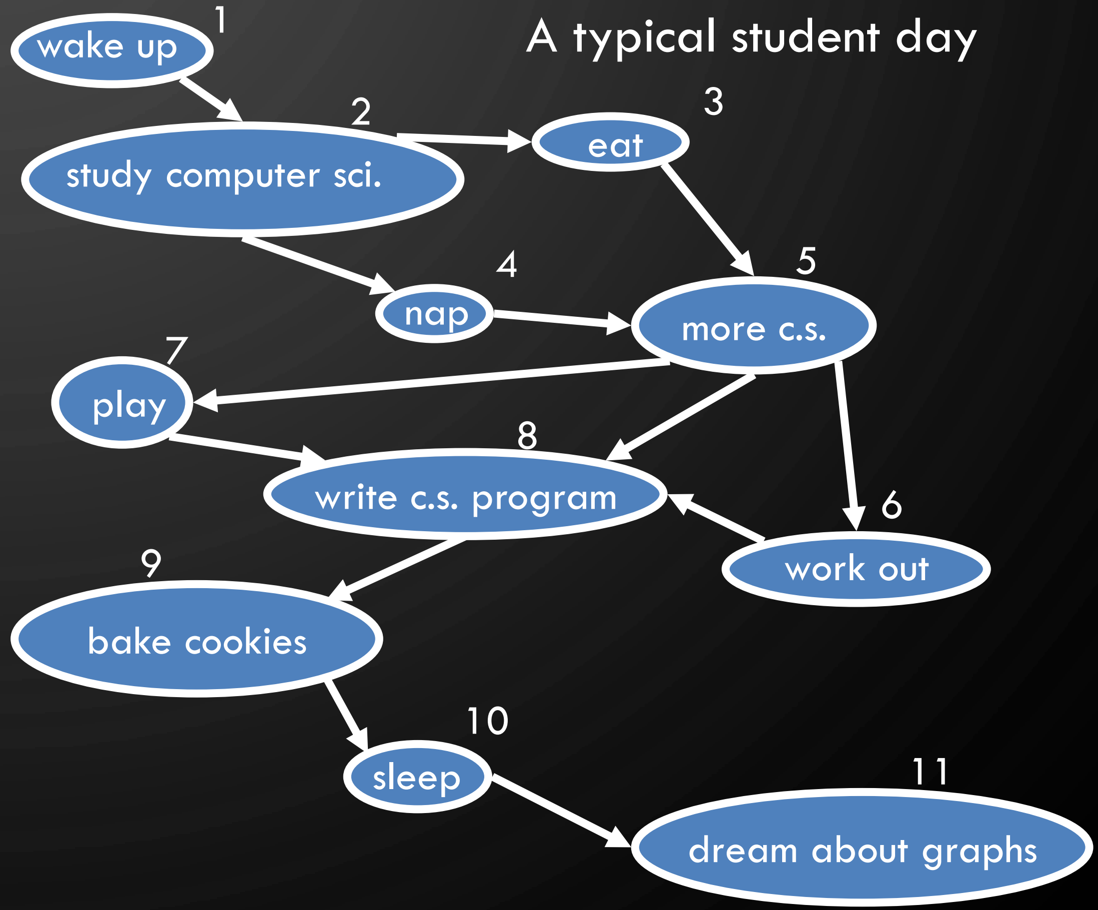
- Number vertices, so that (u, v) in E implies $u < v$





EXERCISE
TOPOLOGICAL SORTING

- Number vertices, so that (u, v) in E implies $u < v$



ALGORITHM FOR TOPOLOGICAL SORTING

Algorithm TopologicalSort(G)

1. $H \leftarrow G$
2. $n \leftarrow G$.numVertices()
3. **while** $\neg H$.isEmpty() **do**
4. Let v be a vertex with no outgoing edges
5. Label $v \leftarrow n$
6. $n \leftarrow n - 1$
7. H.removeVertex(v)

IMPLEMENTATION WITH DFS

- Simulate the algorithm by using depth-first search
- $O(n+m)$ time.

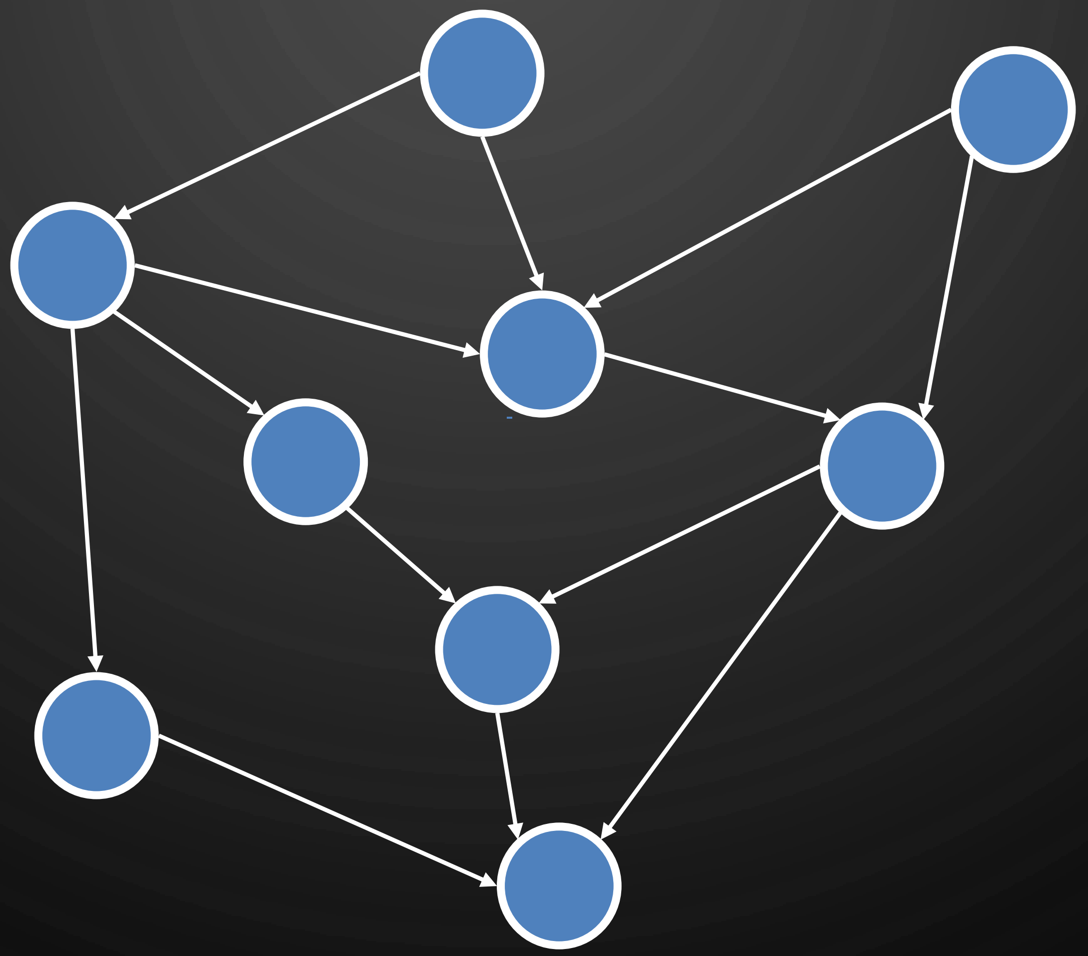
Algorithm topologicalDFS(G)
Input: DAG G
Output: Topological ordering of g

$n \leftarrow G$.numVertices()
Initialize all vertices as *UNEXPLORED*
for each vertex $v \in G$.vertices() **do**
  if getLabel(v) = *UNEXPLORED* **then**
    topologicalDFS(G, v)

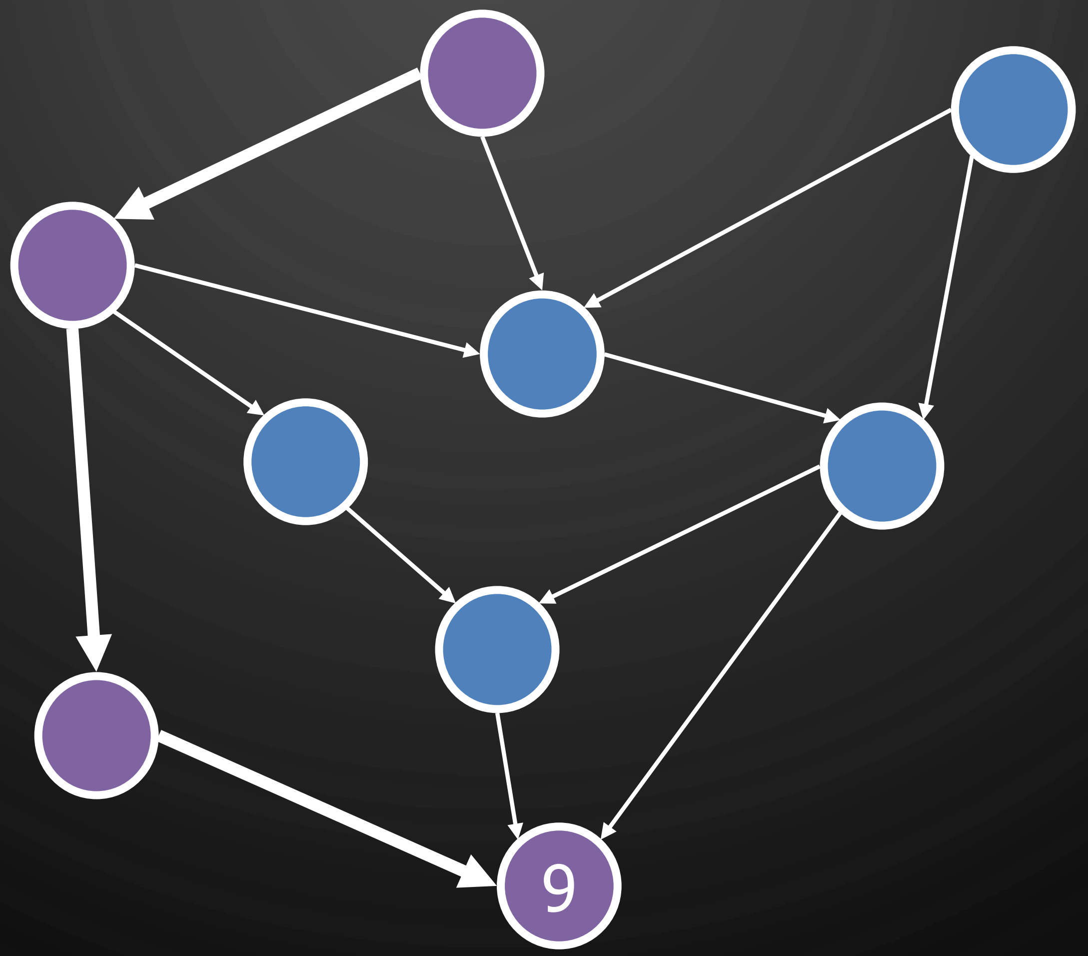
Algorithm topologicalDFS(G, v)
Input: DAG G, start vertex v
Output: Labeling of the vertices of G in the connected component of v

setLabel(v, *VISITED*)
for each $e \in G$.outgoingEdges(v) **do**
  $w \leftarrow G$.opposite(v, e)
  if getLabel(w) = *UNEXPLORED* **then**
    //e is a discovery edge
    topologicalDFS(G, w)
  else
    //e is a forward, cross, or back edge
Label v with topological number n
$n \leftarrow n - 1$

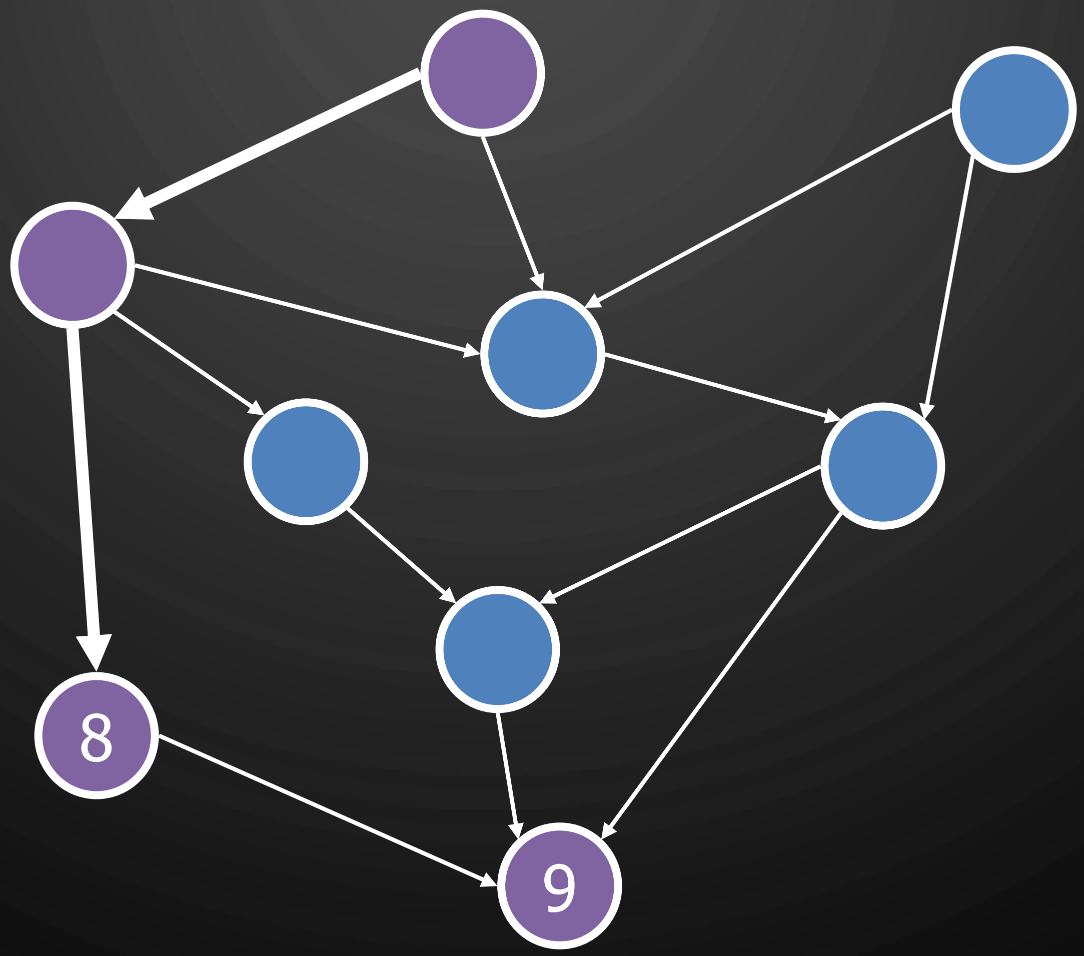
TOPOLOGICAL SORTING EXAMPLE



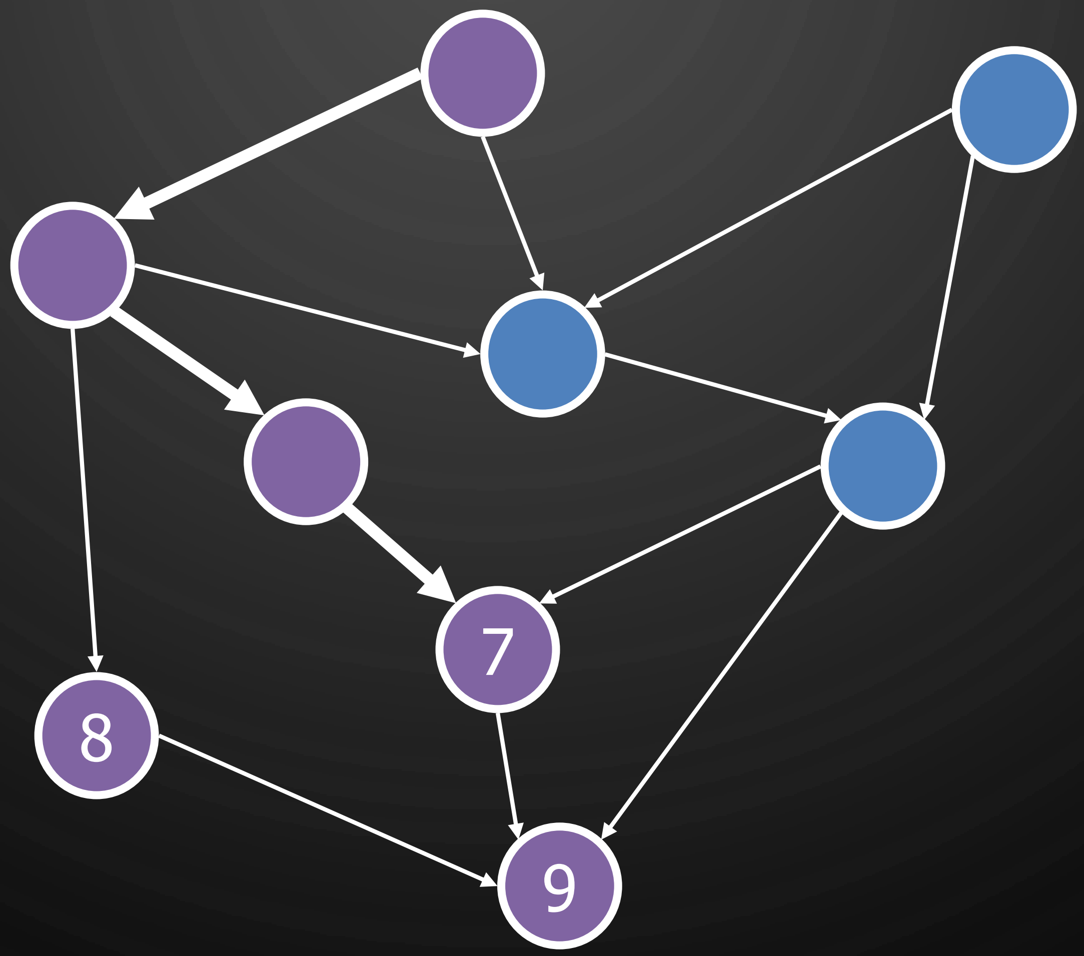
TOPOLOGICAL SORTING EXAMPLE



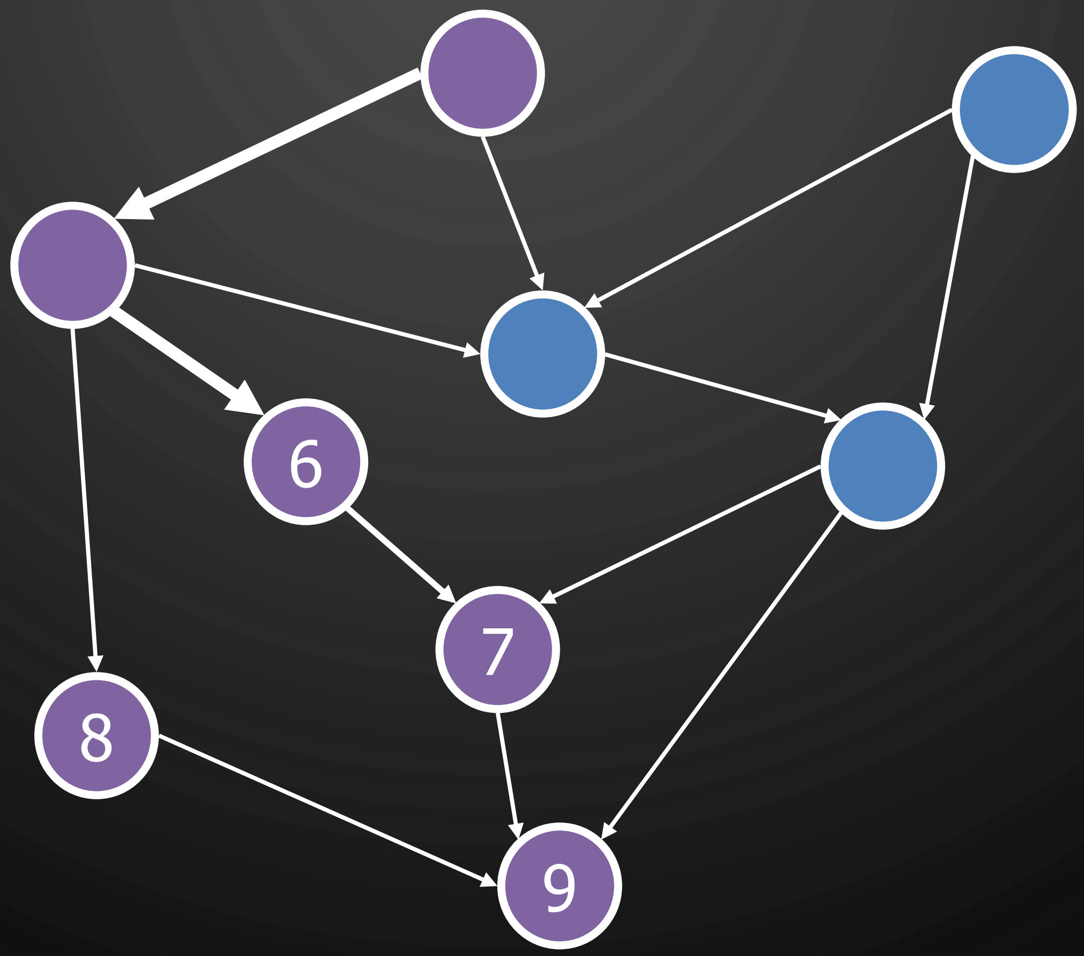
TOPOLOGICAL SORTING EXAMPLE



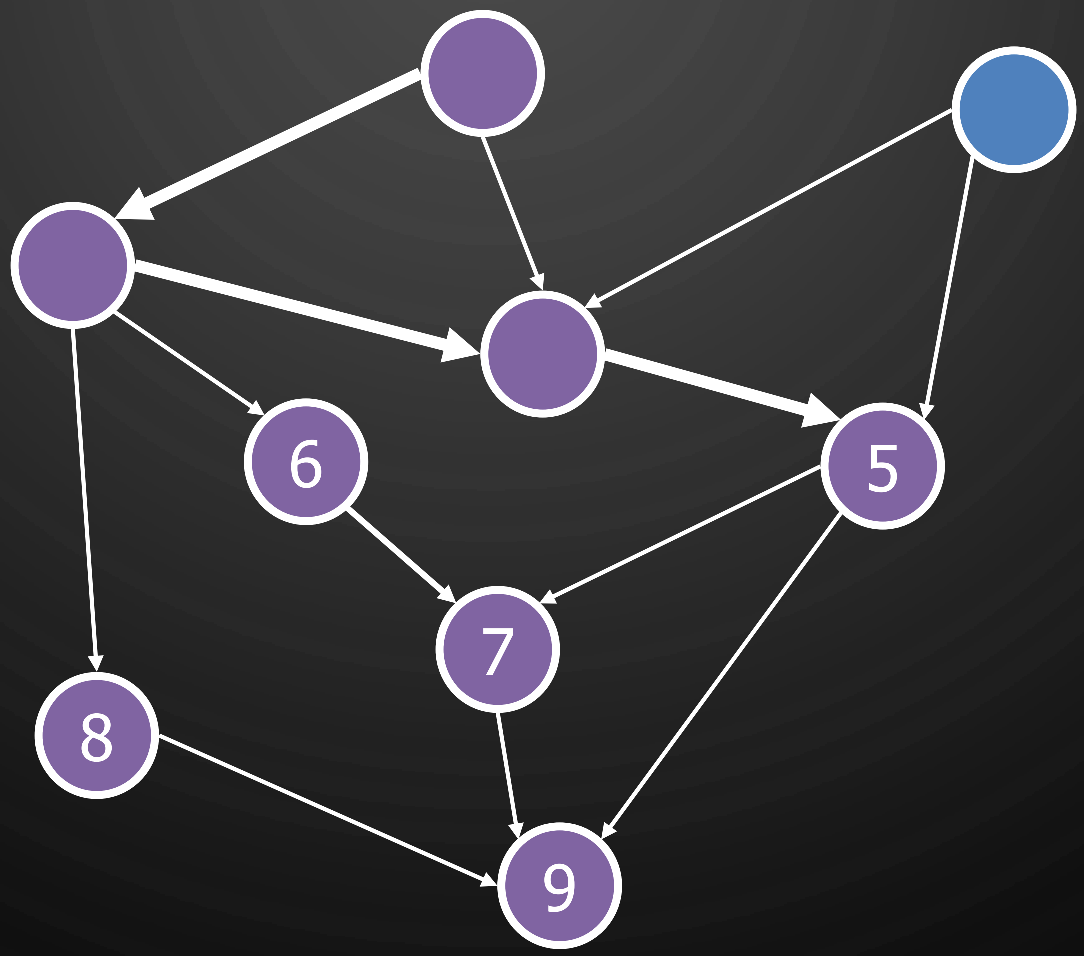
TOPOLOGICAL SORTING EXAMPLE



TOPOLOGICAL SORTING EXAMPLE



TOPOLOGICAL SORTING EXAMPLE



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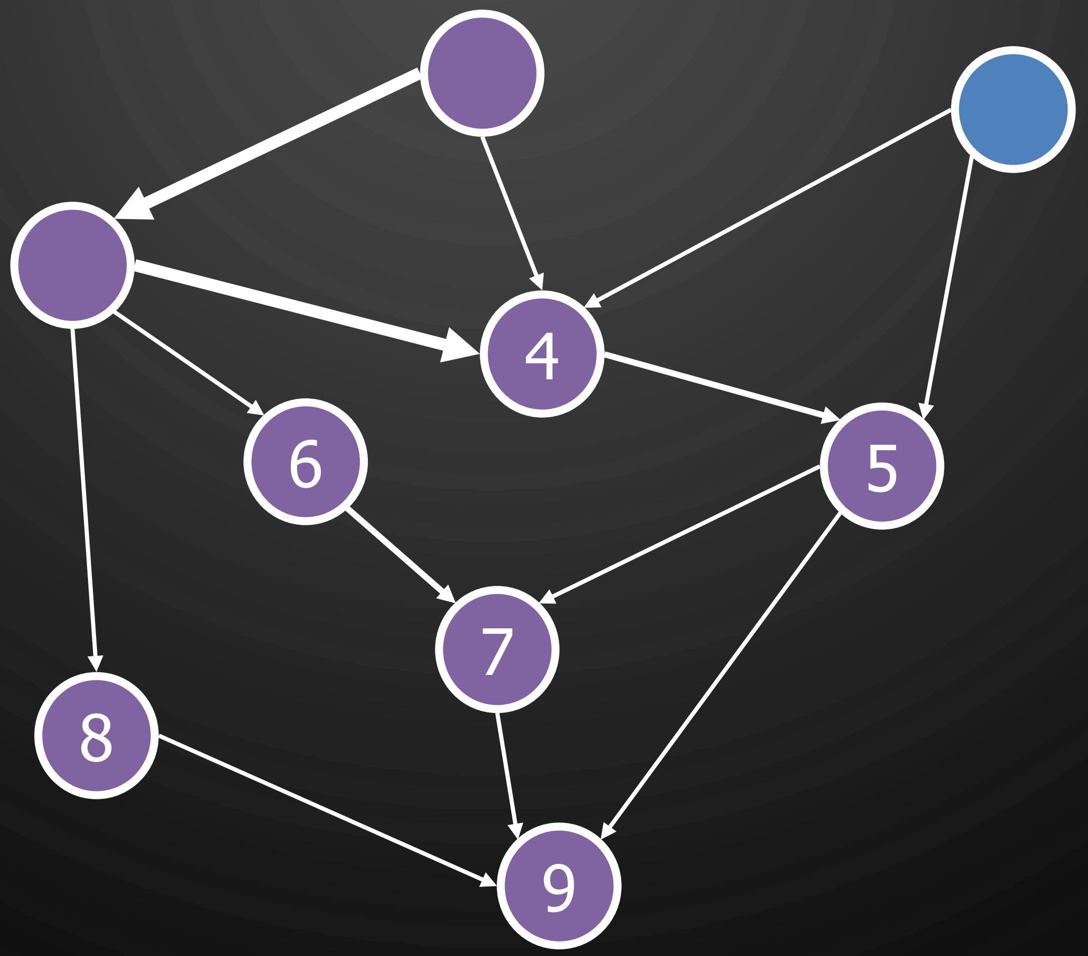

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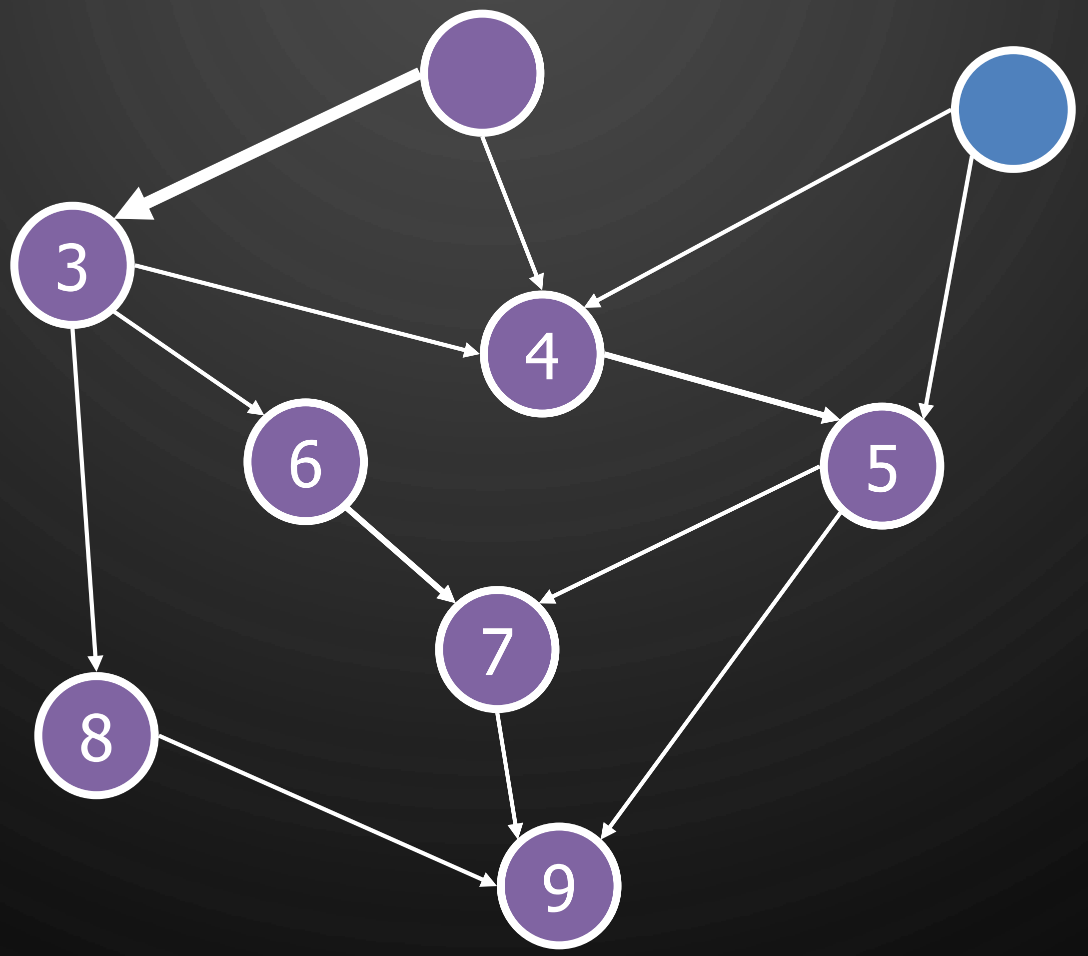
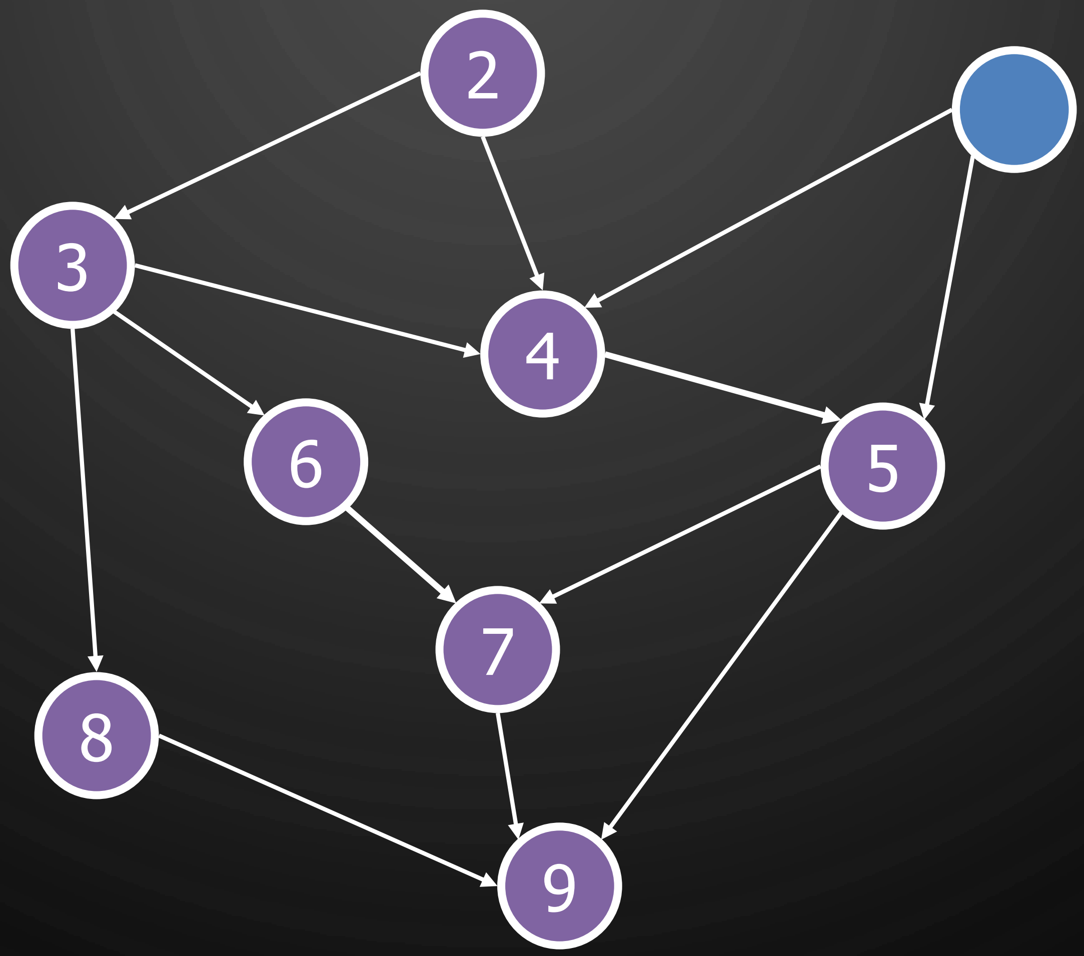


TOPOLOGICAL SORTING EXAMPLE



TOPOLOGICAL SORTING EXAMPLE