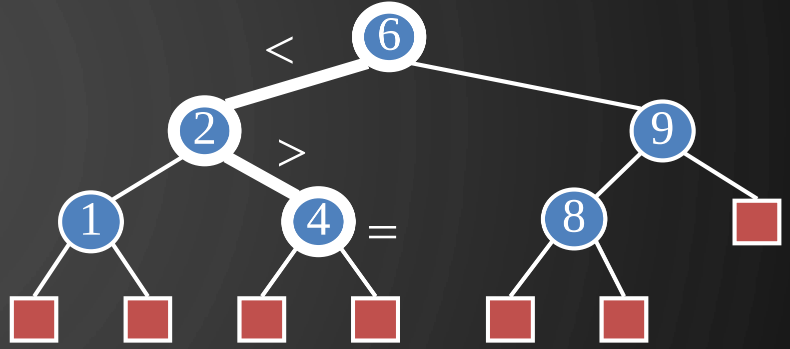


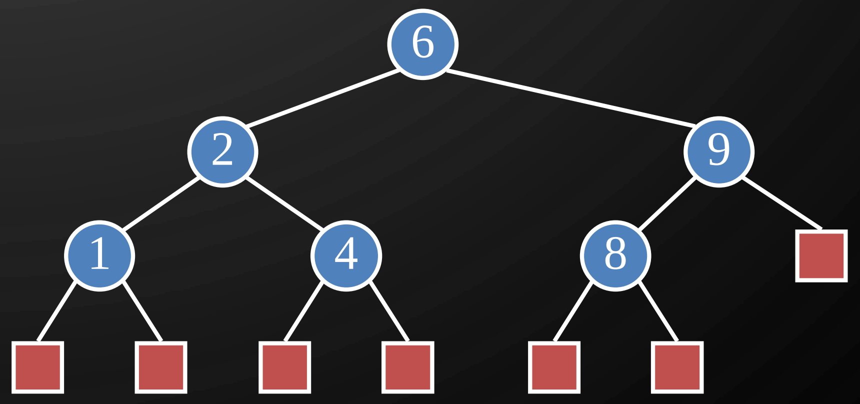
CHAPTER 11
SEARCH TREES

ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN JAVA, GOODRICH, TAMASSIA AND GOLDWASSER (WILEY 2016)

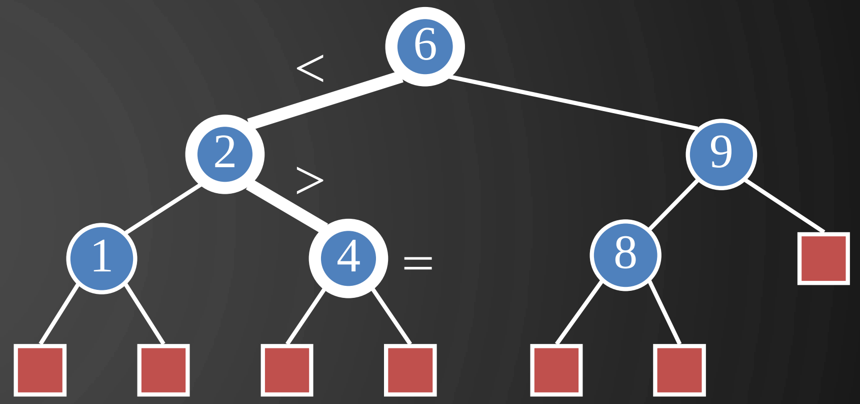
BINARY SEARCH TREES

- A binary search tree is a binary tree storing entries (k, e) (i.e., key-value pairs) at its internal nodes and satisfying the following property:
  - Let u, v, and w be three nodes such that u is in the left subtree of v and w is in the right subtree of v. Then $key(u) \leq key(v) \leq key(w)$
- External nodes do not store items

- An inorder traversal of a binary search trees visits the keys in increasing order



SEARCH

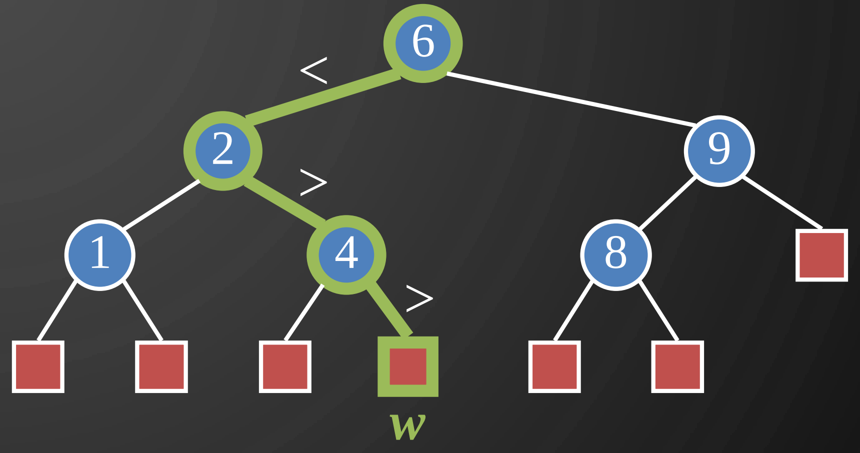


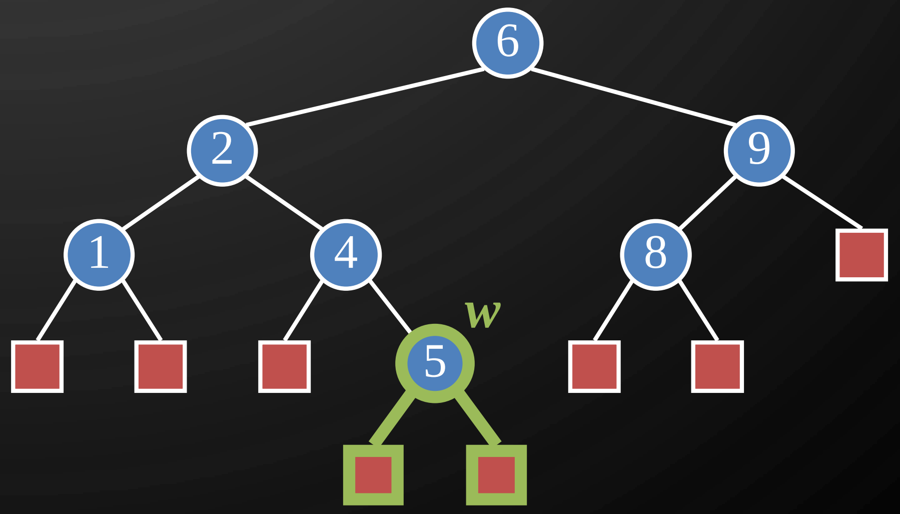
- To search for a key k, we trace a downward path starting at the root
- The next node visited depends on the outcome of the comparison of k with the key of the current node
- If we reach a leaf, the key is not found
- Example: `get(4)`
  - Call `Search(4, root)`
- Algorithms for nearest neighbor queries are similar

```
Algorithm Search(k, v)
Input: Key k, node v
Output: Node with key = k
if v.isExternal()
    return v
if k < v.key()
    return Search(k, v.left())
else if k = v.key()
    return v
else //k > v.key()
    return Search(k, v.right())
```

INSERTION

- To perform operation `put(k, v)`, we search for key k (using `Search(k)`)
- Assume k is not already in the tree, and let let w be the leaf reached by the search
- We insert k at node w and expand w into an internal node
- Example: insert 5



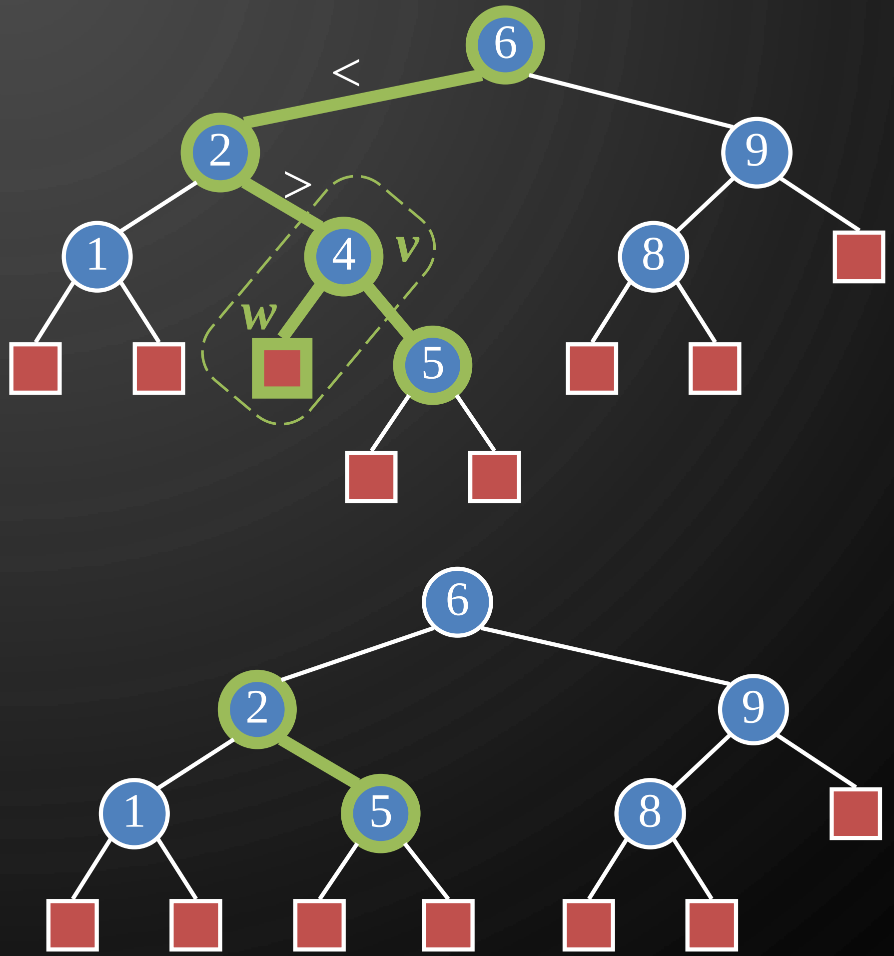


EXERCISE
BINARY SEARCH TREES

- Insert into an initially empty binary search tree items with the following keys (in this order). Draw the resulting binary search tree
  - 30, 40, 24, 58, 48, 26, 11, 13

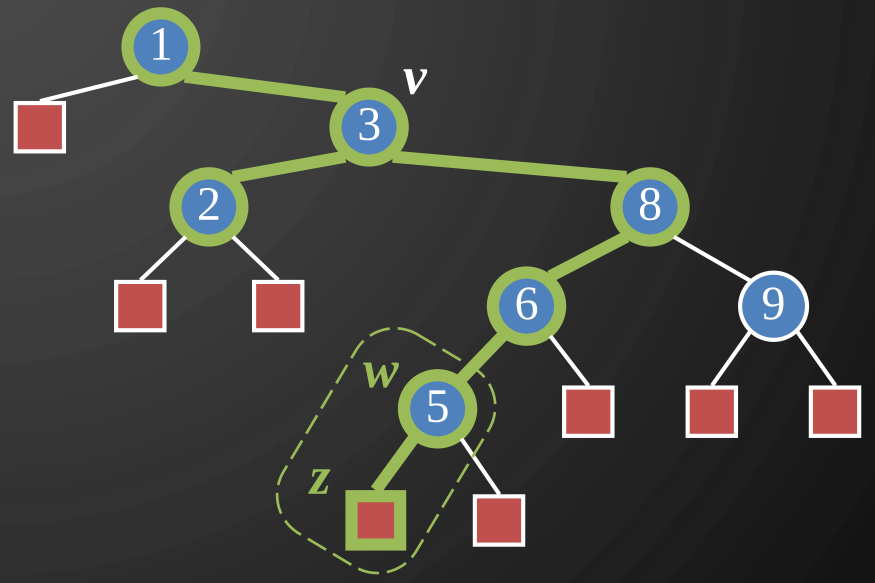
DELETION

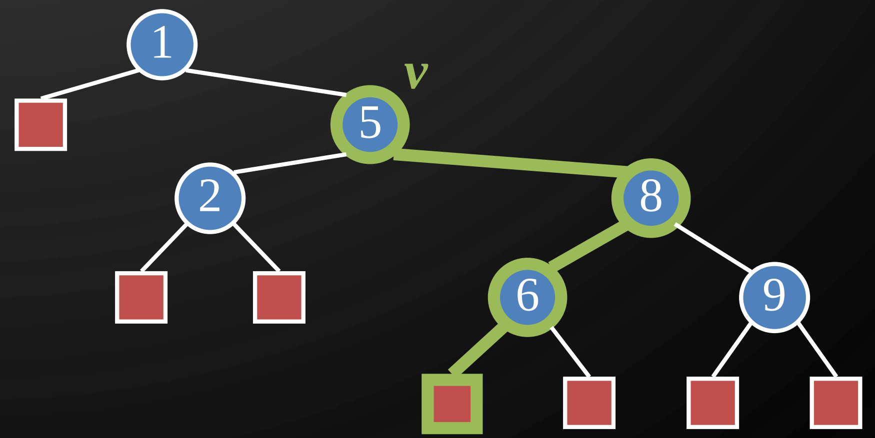
- To perform operation `remove(k)`, we search for key k
- Assume key k is in the tree, and let v be the node storing k
- If node v has a leaf child w, we remove v and w from the tree with operation `removeExternal(w)`, which removes w and its parent
- Example: remove 4



DELETION (CONT.)

- We consider the case where the key k to be removed is stored at a node v whose children are both internal
  - we find the internal node w that follows v in an inorder traversal
  - we copy w.`key()` into node v
  - we remove node w and its left child z (which must be a leaf) by means of operation `removeExternal(z)`
- Example: remove 3



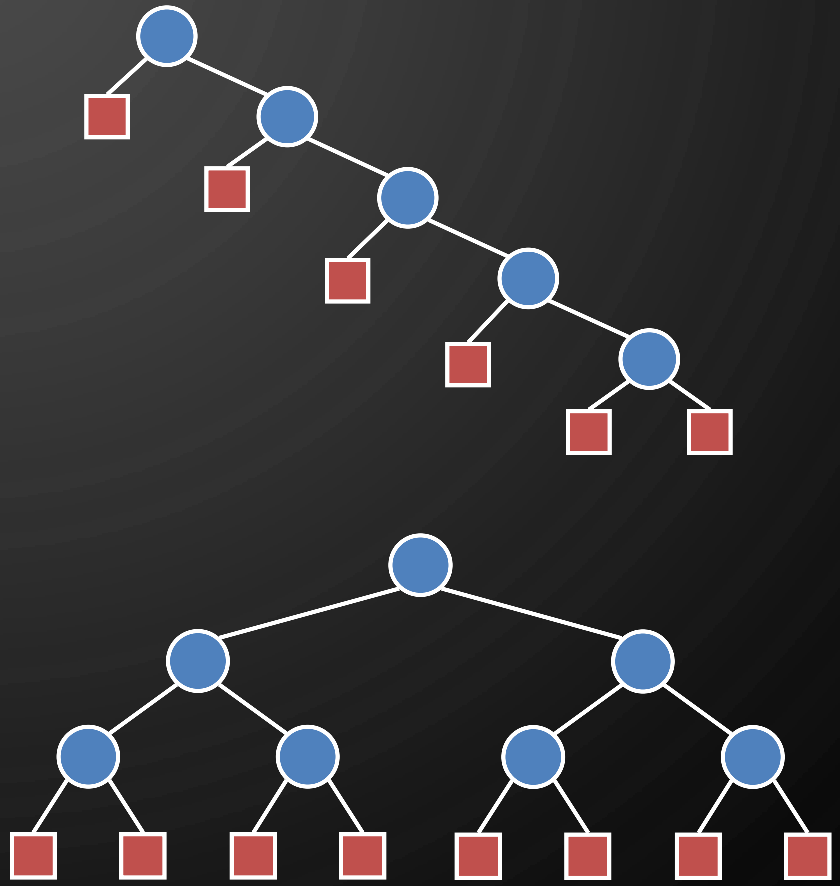


EXERCISE
BINARY SEARCH TREES

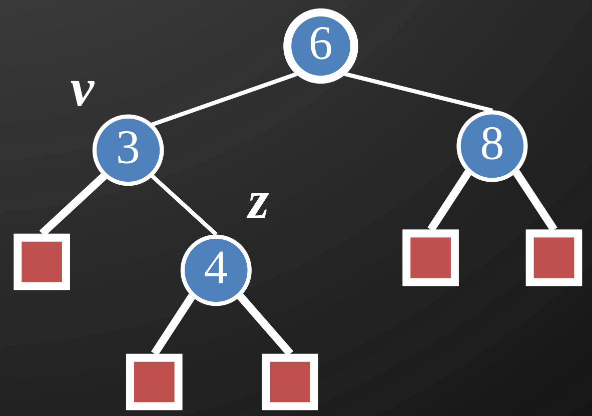
- Insert into an initially empty binary search tree items with the following keys (in this order). Draw the resulting binary search tree
  - 30, 40, 24, 58, 48, 26, 11, 13
- Now, remove the item with key 30. Draw the resulting tree
- Now remove the item with key 48. Draw the resulting tree.

PERFORMANCE

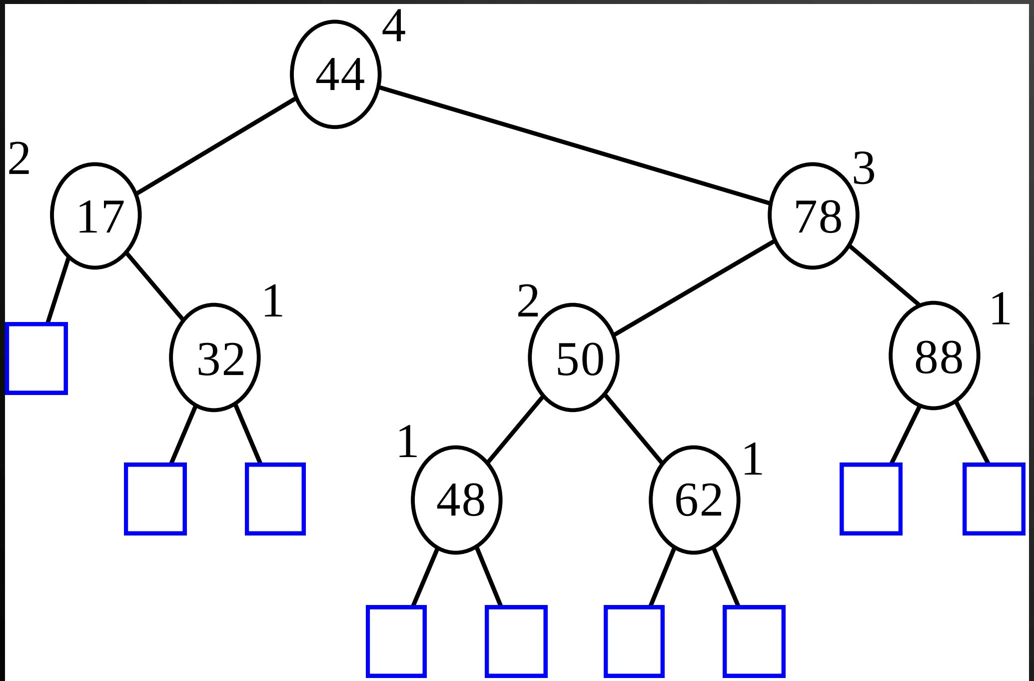
- Consider an ordered map with n items implemented by means of a binary search tree of height h
  - Space used is $O(n)$
  - Methods $get(k)$, `put(k, v)`, and `remove(k)` take $O(h)$ time
- The height h is $O(n)$ in the worst case and $O(\log n)$ in the best case



AVL TREES



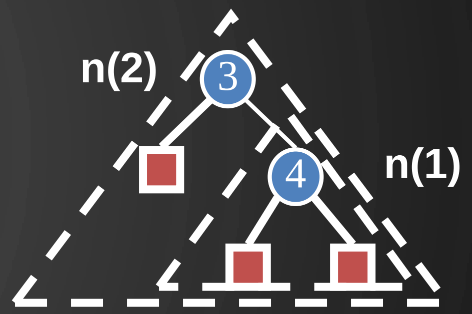
AVL TREE DEFINITION



An example of an AVL tree where the heights are shown next to the nodes:

- AVL trees are balanced
- An AVL Tree is a binary search tree such that for every internal node v of T, the heights of the children of v can differ by at most 1

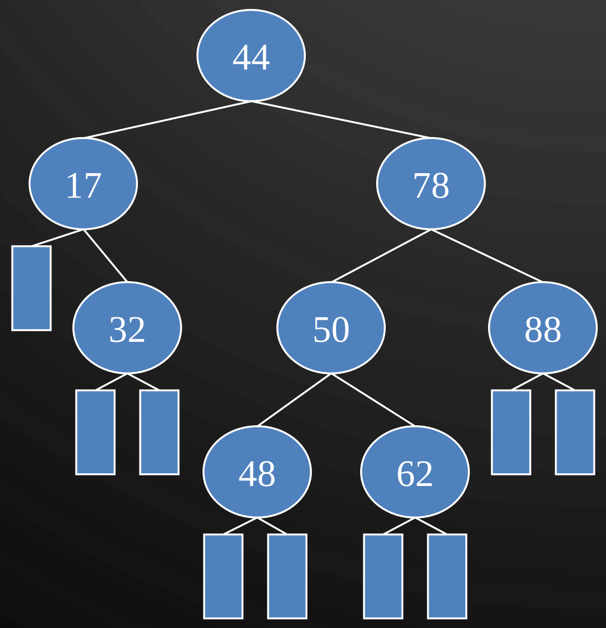
HEIGHT OF AN AVL TREE



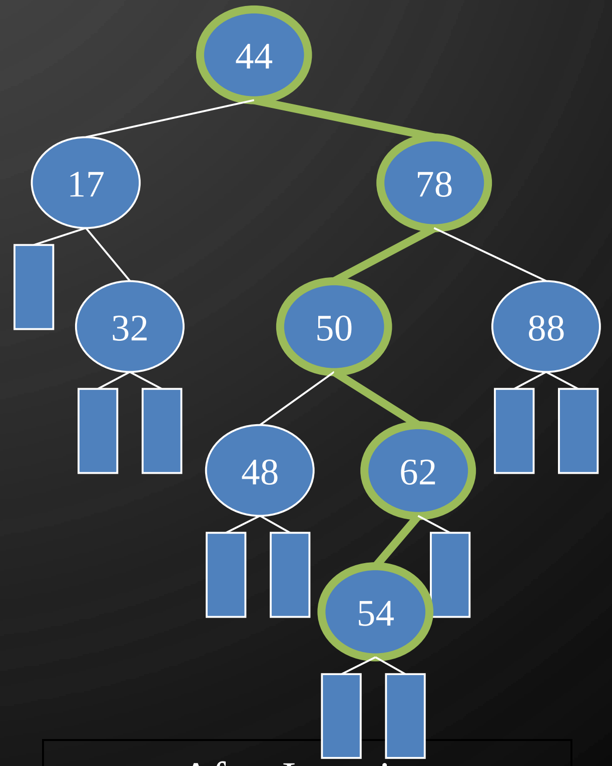
- Fact: The height of an AVL tree storing n keys is $O(\log n)$.
- Proof: Let us bound $n(h)$: the minimum number of internal nodes of an AVL tree of height h.
- We easily see that $n(1) = 1$ and $n(2) = 2$
- For $n > 2$, an AVL tree of height h contains the root node, one AVL subtree of height $h-1$ and another of height $h-2$.
- That is, $n(h) = 1 + n(h-1) + n(h-2)$
- Knowing $n(h-1) > n(h-2)$, we get $n(h) > 2n(h-2)$. So
  - $n(h) > 2n(h-2) > 4n(h-4) > 8n(n-6), \ldots$ (by induction),
  - $n(h) > 2^i n(h-2i)$
- Solving the base case we get: $n(h) > 2^{\frac{h}{2}-1}$
- Taking logarithms: $h < 2\log n(h) + 2$
- Thus the height of an AVL tree is $O(\log n)$

INSERTION IN AN AVL TREE

- Insertion is as in a binary search tree
- Always done by expanding an external node.
- Example insert 54:



Before Insertion

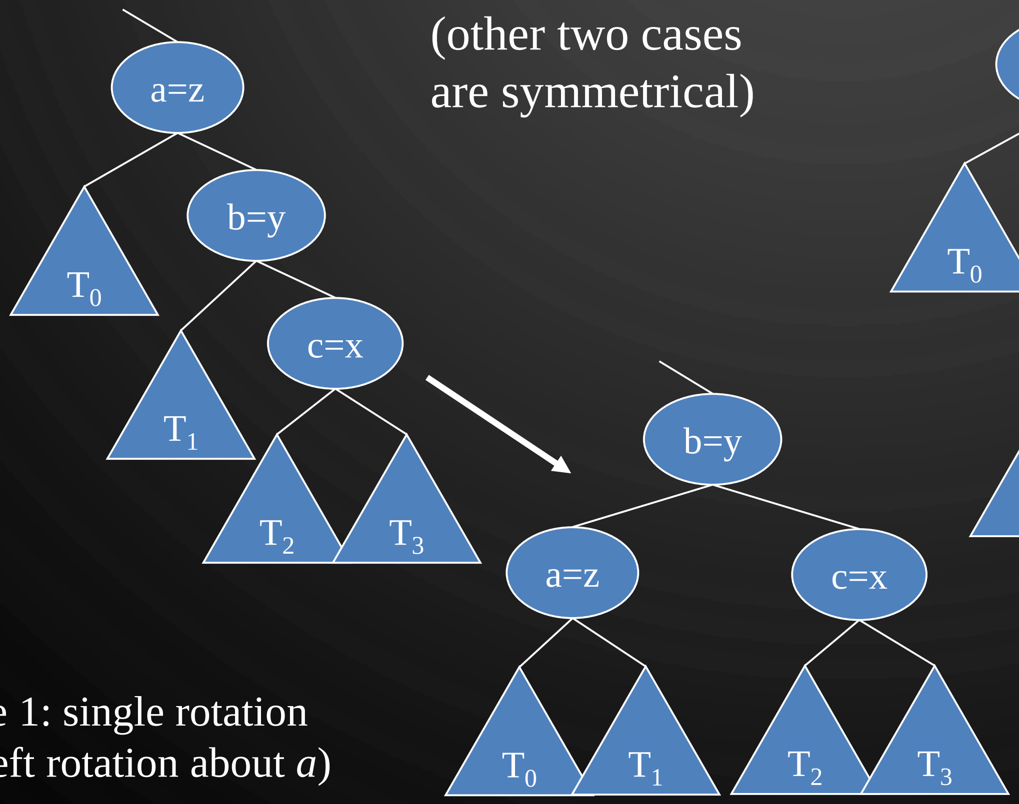


After Insertion

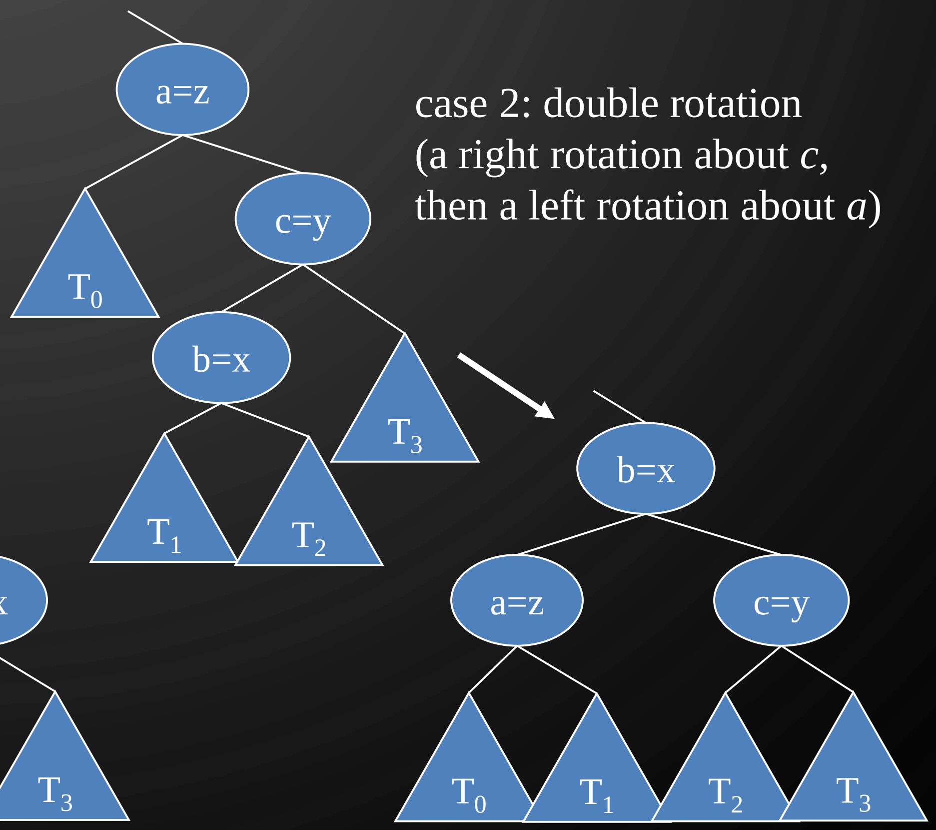
TRINODE RESTRUCTURING

- let (a, b, c) be an inorder listing of x, y, z
- perform the rotations needed to make b the topmost node of the three

(other two cases are symmetrical)

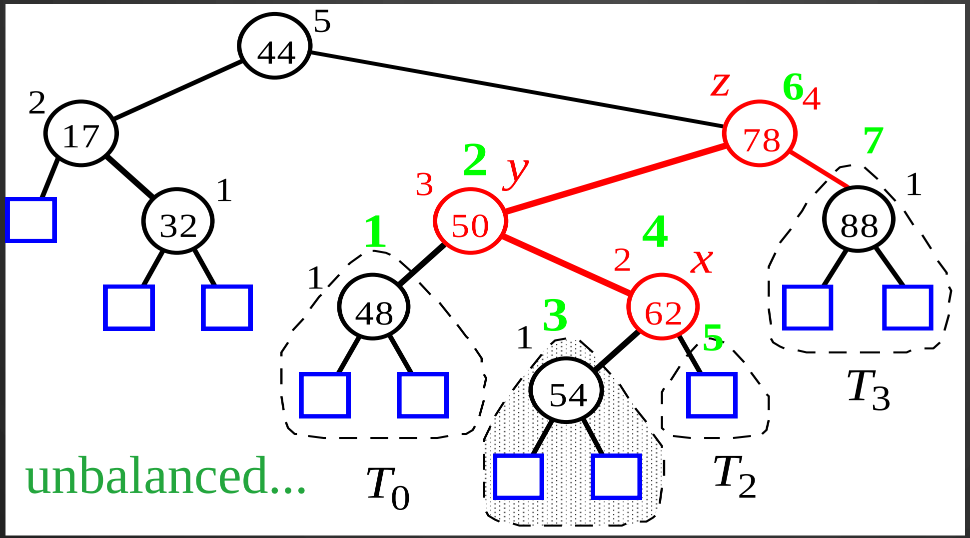


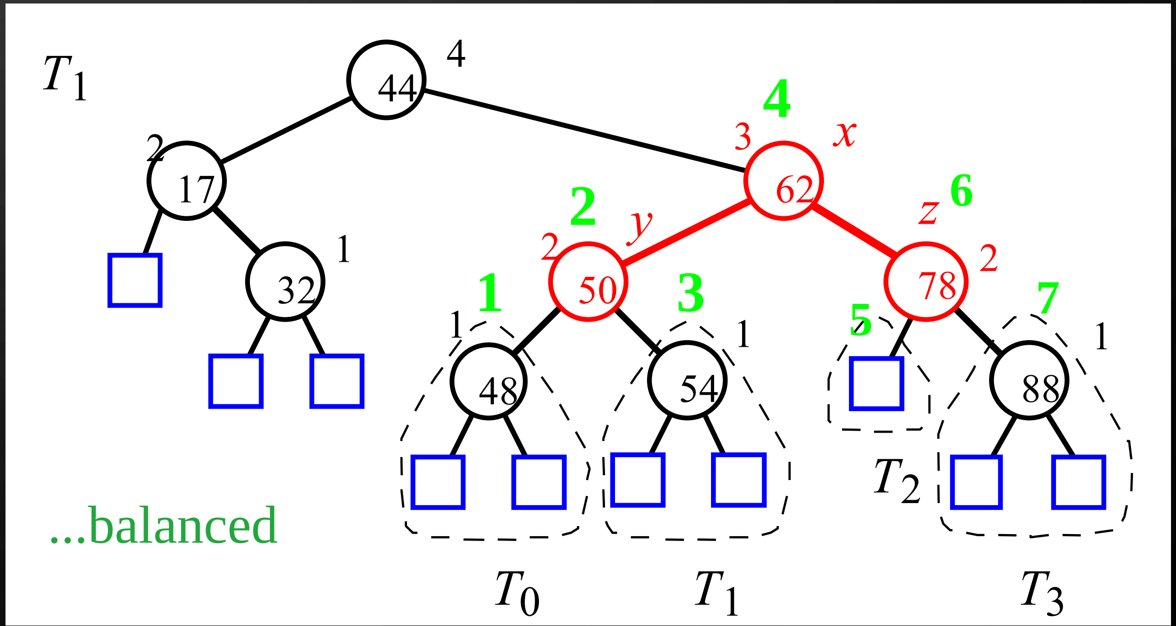
case 1: single rotation
(a left rotation about *a*)



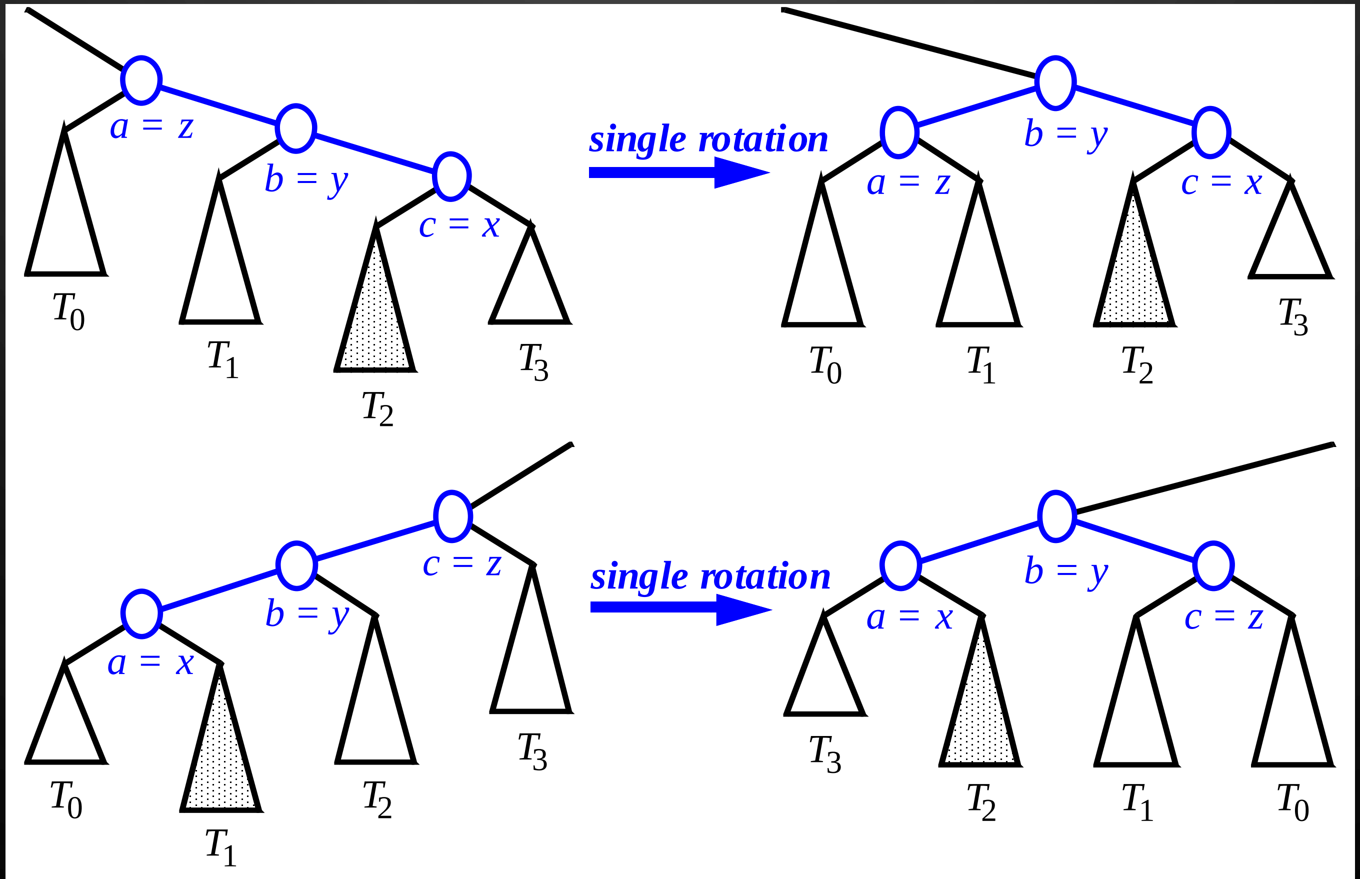
case 2: double rotation
(a right rotation about *c*,
then a left rotation about *a*)

INSERTION EXAMPLE, CONTINUED

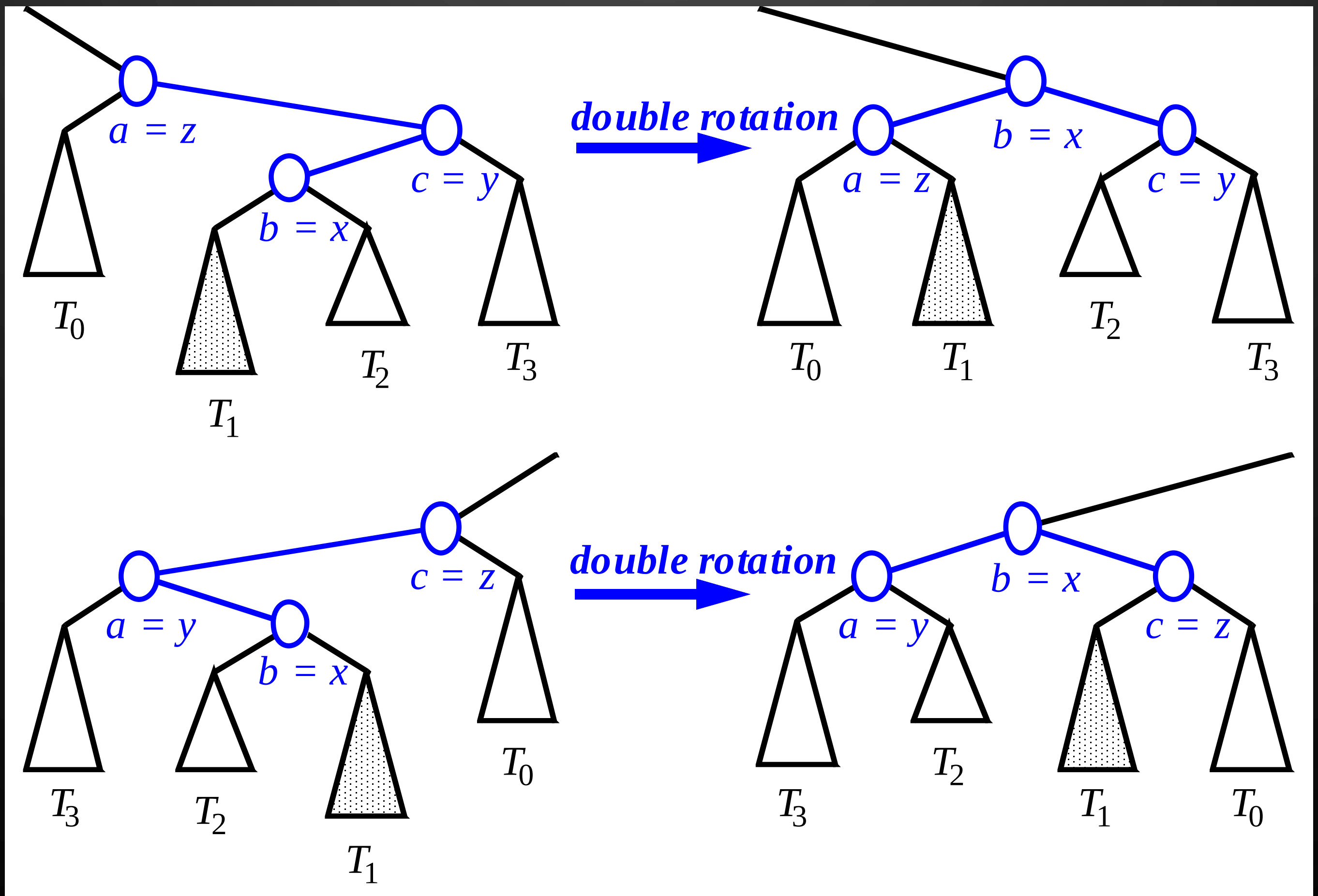




RESTRUCTURING
SINGLE ROTATIONS



RESTRUCTURING
DOUBLE ROTATIONS

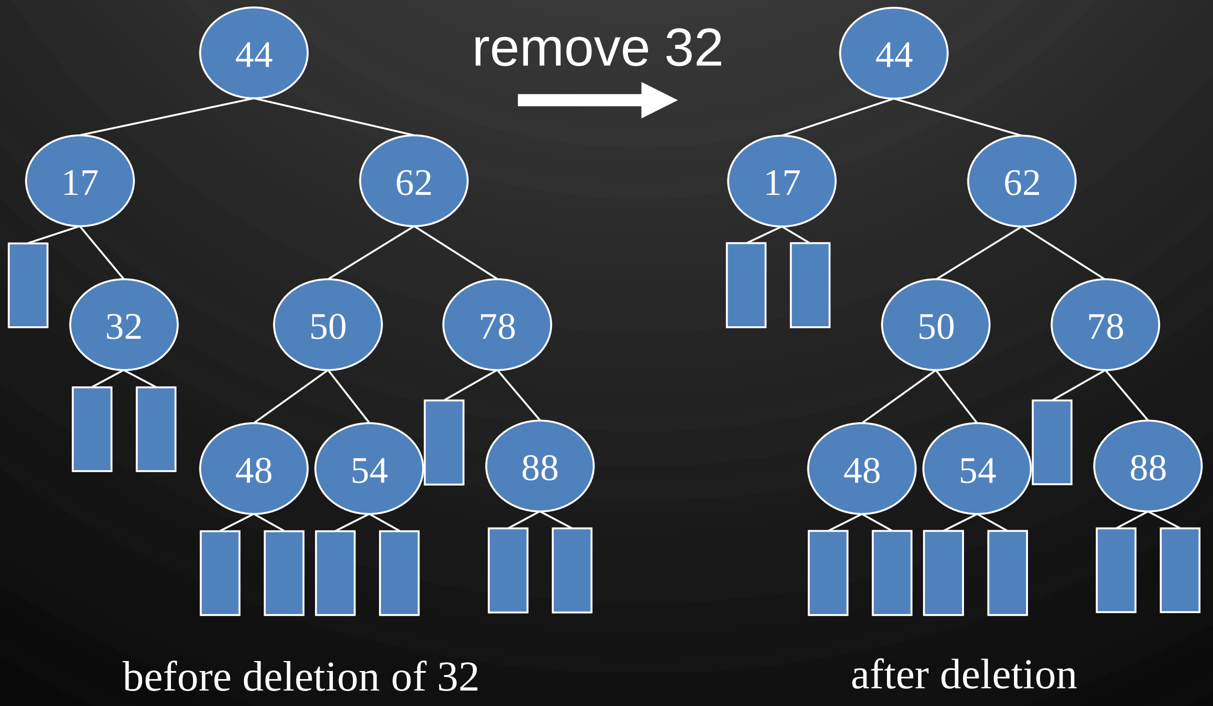


EXERCISE
AVL TREES

- Insert into an initially empty AVL tree items with the following keys (in this order). Draw the resulting AVL tree
  - 30, 40, 24, 58, 48, 26, 11, 13

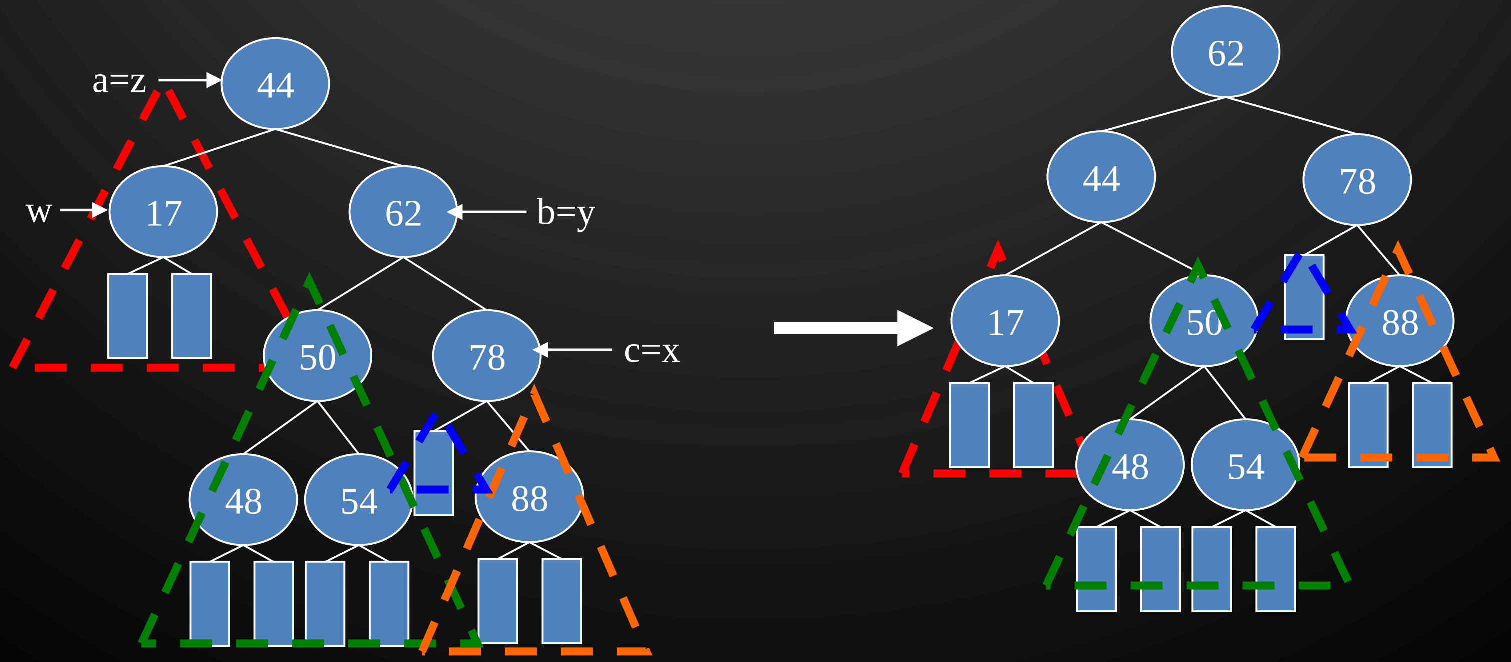
REMOVAL IN AN AVL TREE

- Removal begins as in a binary search tree, which means the node removed will become an empty external node. Its parent, w, may cause an imbalance.
- Example:



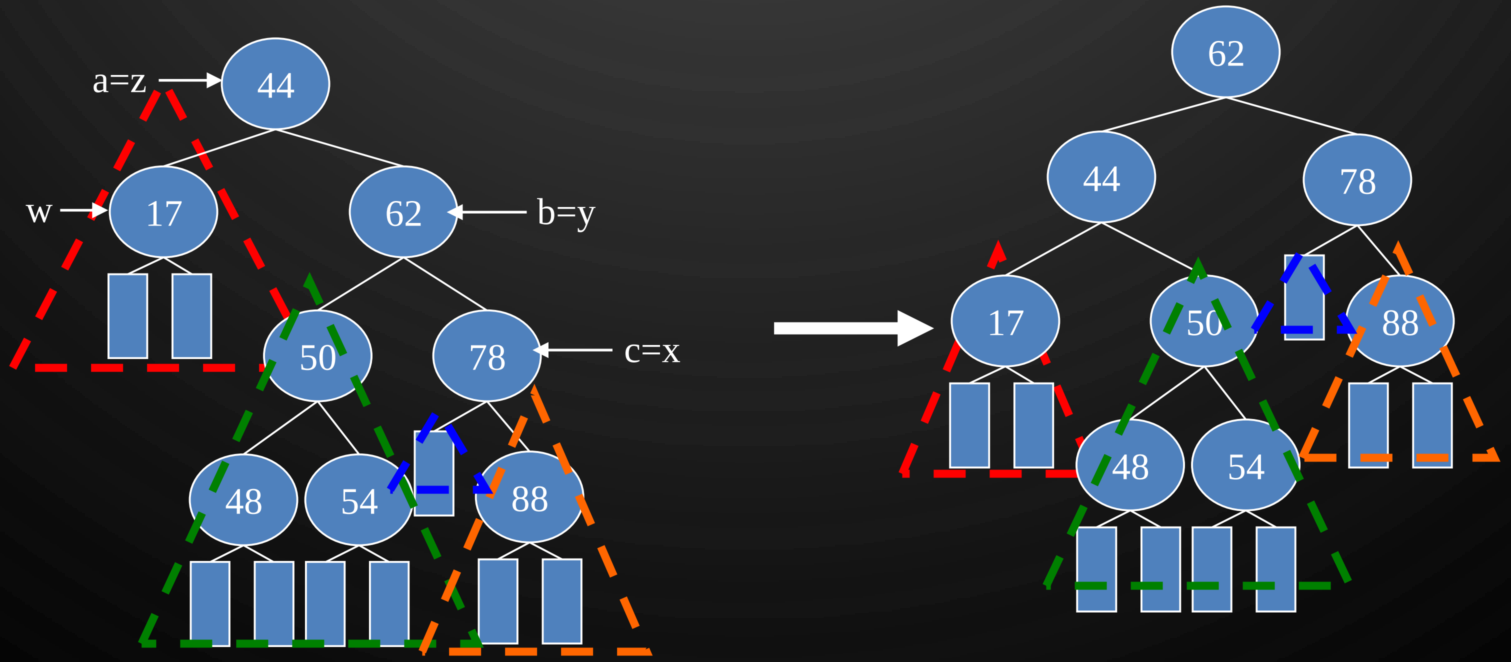
REBALANCING AFTER A REMOVAL

- Let z be the first unbalanced node encountered while travelling up the tree from w (parent of removed node) . Also, let y be the child of z with the larger height, and let x be the child of y with the larger height.
- We perform `restructure(x)` to restore balance at z.



REBALANCING AFTER A REMOVAL

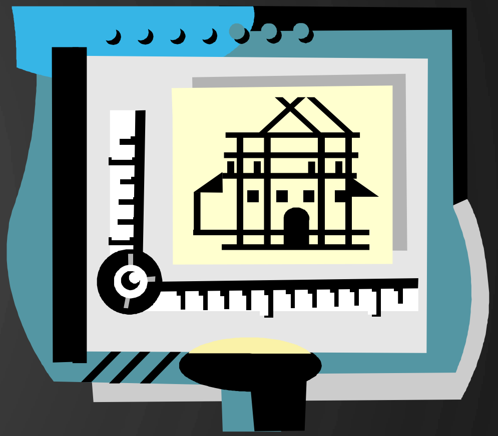
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached
  - This can happen at most $O(\log n)$ times. Why?



EXERCISE
AVL TREES

- Insert into an initially empty AVL tree items with the following keys (in this order). Draw the resulting AVL tree
  - 30, 40, 24, 58, 48, 26, 11, 13
- Now, remove the item with key 48. Draw the resulting tree
- Now, remove the item with key 58. Draw the resulting tree

RUNNING TIMES FOR AVL TREES



- A single restructure is $O(1)$ – using a linked-structure binary tree
- `get(k)` takes $O(\log n)$ time – height of tree is $O(\log n)$, no restructures needed
- `put(k, v)` takes $O(\log n)$ time
    - Initial find is $O(\log n)$
    - Restructuring up the tree, maintaining heights is $O(\log n)$
- `remove(k)` takes $O(\log n)$ time
    - Initial find is $O(\log n)$
    - Restructuring up the tree, maintaining heights is $O(\log n)$

OTHER TYPES OF SELF-BALANCING TREES

- **Splay Trees** – A binary search tree which uses an operation $\mathrm{splay}(x)$ to allow for amortized complexity of $O(\log n)$
- **$(2,4)$ Trees** – A multiway search tree where every node stores internally a list of entries and has 2, 3, or 4 children. Defines self-balancing operations
- **Red-Black Trees** – A binary search tree which colors each internal node red or black. Self-balancing dictates changes of colors and required rotation operations