CHAPTER 14
GRAPH ALGORITHMS

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DEPTH-FIRST SEARCH
DEPTH-FIRST SEARCH

- **Depth-first search (DFS)** is a general technique for traversing a graph.
- A DFS traversal of a graph $G$
  - Visits all the vertices and edges of $G$
  - Determines whether $G$ is connected
  - Computes the connected components of $G$
  - Computes a spanning forest of $G$

- DFS on a graph with $n$ vertices and $m$ edges takes $O(n + m)$ time.
- DFS can be further extended to solve other graph problems:
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs as what Euler tour is to binary trees.
DFS ALGORITHM FROM A VERTEX

**Algorithm** DFS($G, u$)

**Input:** A graph $G$ and a vertex $u$ of $G$

**Output:** A collection of vertices reachable from $u$, with their discovery edges

1. Mark $u$ as visited
2. for each edge $e = (u, v) \in G$.outgoingEdges($u$) do
   3. if $v$ has not been visited then
   4. Record $e$ as a discovery edge for $v$
   5. DFS($G, v$)
EXAMPLE

unexplored vertex
visited vertex
unexplored edge
discovery edge
back edge

\[ I(A) = \{B, C, D, E\} \]

\[ I(B) = \{A, C, F\} \]
\[ I(B) = \{A, C, F\} \]

\[ I(C) = \{A, B, D, E\} \]
\[ I(C) = \{A, B, D, E\} \]

\[ I(C) = \{A, B, D, E\} \]
\[ I(C) = \{A, B, D, E\} \]
\[ I(C) = \{A, B, D, E\} \]

\[ I(D) = \{A, C\} \]

\[ I(E) = \{A, C\} \]

\[ I(C) = \{A, B, D, E\} \]

\[ I(D) = \{A, C\} \]

\[ I(E) = \{A, C\} \]
$I(C) = \{A, B, D, E\}$

$I(B) = \{A, C, F\}$

$I(F) = \{B\}$

$I(B) = \{A, C, F\}$

$I(A) = \{A, B, C, D\}$

$I(G) = \emptyset$
EXERCISE
DFS ALGORITHM

• Perform DFS of the following graph, start from vertex A
  • Assume adjacent edges are processed in alphabetical order
  • Number vertices in the order they are visited
  • Label edges as discovery or back edges
DFS AND MAZE TRAVERSAL

• The DFS algorithm is similar to a classic strategy for exploring a maze
  • We mark each intersection, corner and dead end (vertex) visited
  • We mark each corridor (edge) traversed
  • We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)
The algorithm uses a mechanism for setting and getting "labels" of vertices and edges.

**Algorithm DFS(G)**

**Input:** Graph G

**Output:** Labeling of the edges of G as discovery edges and back edges

1. for each \( v \in G \text{.vertices()} \) do
2. \( \text{setLabel}(v, \text{UNEXPLORED}) \)
3. for each \( e \in G\text{.edges()} \) do
4. \( \text{setLabel}(e, \text{UNEXPLORED}) \)
5. for each \( v \in G\text{.vertices()} \) do
6. if getLabel\((v) = \text{UNEXPLORED} \) then
7. DFS\((G,v)\)

**Algorithm DFS(G,v)**

**Input:** Graph G and a start vertex v

**Output:** Labeling of the edges of G in the connected component of v as discovery edges and back edges

1. \( \text{setLabel}(v, \text{VISITED}) \)
2. for each \( e \in G\text{.outgoingEdges}(v) \) do
3. if getLabel\((e) = \text{UNEXPLORED} \) then
4. \( w \leftarrow G\text{.opposite}(v,e) \)
5. if getLabel\((w) = \text{UNEXPLORED} \) then
6. \( \text{setLabel}(e, \text{DISCOVERY}) \)
7. DFS\((G,w)\)
8. else
9. \( \text{setLabel}(e, \text{BACK}) \)
PROPERTIES OF DFS

• Property 1
  • $\text{DFS}(G, v)$ visits all the vertices and edges in the connected component of $v$

• Property 2
  • The discovery edges labeled by $\text{DFS}(G, v)$ form a spanning tree of the connected component of $v$
ANALYSIS OF DFS

• Setting/getting a vertex/edge label takes $O(1)$ time

• Each vertex is labeled twice
  • once as UNEXPLORED
  • once as VISITED

• Each edge is labeled twice
  • once as UNEXPLORED
  • once as DISCOVERY or BACK

• Function DFS($G, v$) and the method outgoingEdges() are called once for each vertex

• DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  • Recall that $\Sigma_v \deg(v) = 2m$
APPLICATION
PATH FINDING

- We can specialize the DFS algorithm to find a path between two given vertices \( u \) and \( z \) using the template method pattern
- We call DFS\((G, u)\) with \( u \) as the start vertex
- We use a stack \( S \) to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex \( z \) is encountered, we return the path as the contents of the stack

**Algorithm** pathDFS\((G, v, z)\)

**Input:** Graph \( G \), a start vertex \( v \), a goal vertex \( z \)

**Output:** Path between \( v \) and \( z \)

1. setLabel\((v, VISITED)\)
2. \( S.push(v) \)
3. if \( v = z \) then
4. return \( S.elements() \)
5. for each \( e \in G.outgoingEdges(v) \) do
6. if getLabel\((e) = UNEXPLORED\) then
7. \( w \leftarrow G.opposite(v, e) \)
8. if getLabel\((w) = UNEXPLORED \) then
9. setLabel\((e, DISCOVERY)\)
10. \( S.push(e) \)
11. pathDFS\((G, w)\)
12. \( S.pop() \)
13. else
14. setLabel\((e, BACK)\)
15. \( S.pop() \)
APPLICATION
CYCLE FINDING

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack $S$ to keep track of the path between the start vertex and the current vertex
- As soon as a back edge $(v, w)$ is encountered, we return the cycle as the portion of the stack from the top to vertex $w$

Algorithm cycleDFS($G, v$)
Input: Graph $G$, a start vertex $v$
Output: Cycle containing $v$
1. setLabel($v$, VISITED)
2. $S$.push($v$)
3. for each $e \in G$.outgoingEdges($v$) do
4.   if getLabel($e$) = UNEXPLORED then
5.      $w \leftarrow G$.opposite($v, e$)
6.      $S$.push($e$)
7.      if getLabel($w$) = UNEXPLORED then
8.         setLabel($e$, DISCOVERY)
9.         cycleDFS($G, w$)
10. $S$.pop()
11. else
12.    Stack $T \leftarrow \emptyset$
13.    repeat
14.      $T$.push($S$.pop())
15.      until $T$.top() = $w$
16.    return $T$.elements()
17. $S$.pop()
DIRECTED DFS

- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction.
- In the directed DFS algorithm, we have four types of edges:
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting at a vertex $s$ determines the vertices reachable from $s$. 
REACHABILITY

• DFS tree rooted at $v$: vertices reachable from $v$ via directed paths
STRONG CONNECTIVITY

• Each vertex can reach all other vertices
STRONG CONNECTIVITY ALGORITHM

• Pick a vertex \( v \) in \( G \)
• Perform a DFS from \( v \) in \( G \)
  • If there’s a \( w \) not visited, print “no”
• Let \( G' \) be \( G \) with edges reversed
• Perform a DFS from \( v \) in \( G' \)
  • If there’s a \( w \) not visited, print “no”
  • Else, print “yes”
• Running time: \( O(n + m) \)
STRONGLY CONNECTED COMPONENTS

- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in $O(n + m)$ time using DFS, but is more complicated (similar to biconnectivity).

(a, c, g)  
{f, d, e, b}
BREADTH-FIRST SEARCH
BREADTH-FIRST SEARCH

- **Breadth-first search (BFS)** is a general technique for traversing a graph
- A BFS traversal of a graph $G$
  - Visits all the vertices and edges of $G$
  - Determines whether $G$ is connected
  - Computes the connected components of $G$
  - Computes a spanning forest of $G$
  - BFS on a graph with $n$ vertices and $m$ edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one
BFS ALGORITHM

The algorithm uses a mechanism for setting and getting "labels" of vertices and edges.

Algorithm BFS(G)
Input: Graph G
Output: Labeling of the edges and partition of the vertices of G
1. for each v ∈ G.vertices() do
2. setLabel(v, UNEXPLORED)
3. for each e ∈ G.edges() do
4. setLabel(e, UNEXPLORED)
5. for each v ∈ G.vertices() do
6. if getLabel(v) = UNEXPLORED then
7. BFS(G,v)

Algorithm BFS(G,s)
Input: Graph G, a start vertex s
1. List L0 ← {s}
2. setLabel(s, VISITED)
3. i ← 0
4. while ¬L_i.isEmpty() do
5. List L_{i+1} ← ∅
6. for each v ∈ L_i do
7. for each e ∈ G.outgoingEdges(v) do
8. if getLabel(e) = UNEXPLORED then
9. w ← G.opposite(v,e)
10. if getLabel(w) = UNEXPLORED then
11. setLabel(e, DISCOVERY)
12. setLabel(w, VISITED)
13. L_{i+1} ← L_{i+1} ∪ {w}
14. else
15. setLabel(e, CROSS)
16. i ← i + 1
EXAMPLE

- **A**: visited vertex
- **A**: unexplored vertex
- **unexplored edge**: dashed line
- **discovery edge**: solid purple line
- **cross edge**: dotted purple line

Diagram: A tree structure with vertices labeled A, B, C, D, E, and F, and edges connecting them.
EXAMPLE

unexplored vertex
visited vertex
unexplored edge
discovery edge
cross edge

A
B
C
D
E
F

L0
L1
L2
EXAMPLE

- **L_0**: Unexplored vertex
- **L_1**: Visited vertex
- **L_2**: Unexplored edge
- **L_3**: Discovery edge
- **L_4**: Cross edge

The diagram illustrates the exploration of a graph with labeled vertices and edges. The process moves from unexplored vertices (L_0) to visited vertices (L_1) through discovery edges (L_2), encountering unexplored edges (L_3) along the way, and identifying cross edges (L_4) in the progression.
EXERCISE  
BFS ALGORITHM

• Perform BFS of the following graph, start from vertex A
  • Assume adjacent edges are processed in alphabetical order
  • Number vertices in the order they are visited and note the level they are in
  • Label edges as discovery or cross edges
PROPERTIES

• Notation
  • $G_s$: connected component of $s$

• Property 1
  • BFS($G, s$) visits all the vertices and edges of $G_s$

• Property 2
  • The discovery edges labeled by BFS($G, s$) form a spanning tree $T_s$ of $G_s$

• Property 3
  • For each vertex $v \in L_i$
    • The path of $T_s$ from $s$ to $v$ has $i$ edges
    • Every path from $s$ to $v$ in $G_s$ has at least $i$ edges
ANALYSIS

• Setting/getting a vertex/edge label takes $O(1)$ time

• Each vertex is labeled twice
  • once as UNEXPLORED
  • once as VISITED

• Each edge is labeled twice
  • once as UNEXPLORED
  • once as DISCOVERY or CROSS

• Each vertex is inserted once into a sequence $L_i$

• Method outgoingEdges() is called once for each vertex

• BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  • Recall that $\Sigma_v \deg(v) = 2m$
APPLICATIONS

• Using the template method pattern, we can specialize the BFS traversal of a graph $G$ to solve the following problems in $O(n + m)$ time
  • Compute the connected components of $G$
  • Compute a spanning forest of $G$
  • Find a simple cycle in $G$, or report that $G$ is a forest
  • Given two vertices of $G$, find a path in $G$ between them with the minimum number of edges, or report that no such path exists
DFS VS. BFS

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<th>Applications</th>
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<td>Spanning forest, connected components, paths, cycles</td>
<td>√</td>
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<td>Shortest paths</td>
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Applications:
- DFS: Spanning forest, connected components, paths, cycles, Shortest paths, Biconnected components
- BFS: Spanning forest, connected components, paths, cycles, Shortest paths
DFS VS. BFS

Back edge \((v, w)\)

- \(w\) is an ancestor of \(v\) in the tree of discovery edges

Cross edge \((v, w)\)

- \(w\) is in the same level as \(v\) or in the next level in the tree of discovery edges
**DAGS AND TOPOLOGICAL ORDERING**

- A **directed acyclic graph (DAG)** is a digraph that has no directed cycles.
- A topological ordering of a digraph is a numbering:
  - $v_1, ..., v_n$
  - Of the vertices such that for every edge $(v_i, v_j)$, we have $i < j$.
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints.
- Theorem - A digraph admits a topological ordering if and only if it is a DAG.
• Scheduling: edge \((a, b)\) means task \(a\) must be completed before \(b\) can be started
EXERCISE
TOPOLOGICAL SORTING

• Number vertices, so that \((u, v)\) in \(E\) implies \(u < v\)

A typical student day:
- wake up
- eat
- study computer sci.
- nap
- more c.s.
- play
- write c.s. program
- work out
- bake cookies
- sleep
- dream about graphs
EXERCISE
TOPOLOGICAL SORTING

• Number vertices, so that \((u, v)\) in \(E\) implies \(u < v\)
ALGORITHM FOR TOPOLOGICAL SORTING

**Algorithm** TopologicalSort($G$)

**Input:** Directed Acyclic Graph (DAG) $G$

**Output:** Topological ordering of $G$

1. $H \leftarrow G$
2. $n \leftarrow G$.numVertices()
3. while $H$.isEmpty() do
4.   Let $v$ be a vertex with no outgoing edges
5.   Label $v \leftarrow n$
6.   $n \leftarrow n - 1$
7.   $H$.removeVertex($v$)
IMPLEMENTATION WITH DFS

• Simulate the algorithm by using depth-first search
• $O(n + m)$ time.

**Algorithm** topologicalDFS($G$)
*Input*: DAG $G$
*Output*: Topological ordering of $G$
1. $n \leftarrow G$.numVertices()
2. Initialize all vertices as UNEXPLORED
3. for each vertex $v \in G$.vertices() do
4. if getLabel($v$) = UNEXPLORED then
5. topologicalDFS($G,v$)

**Algorithm** topologicalDFS($G,v$)
*Input*: DAG $G$, start vertex $v$
*Output*: Labeling of the vertices of $G$ in the connected component of $v$
1. setLabel($v$, VISITED)
2. for each $e \in G$.outgoingEdges($v$) do
3. $w \leftarrow G$.opposite($v,e$)
4. if getLabel($w$) = UNEXPLORED then
5. // $e$ is a discovery edge
6. topologicalDFS($G,w$)
7. else
8. // $e$ is a forward, cross, or back edge
9. Label $v$ with topological number $n$
10. $n \leftarrow n - 1$
TOPOLOGICAL SORTING EXAMPLE
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MINIMUM SPANNING TREES
MINIMUM SPANNING TREE

• Minimum spanning tree (MST)
  • Spanning tree of a weighted graph with minimum total edge weight

• Applications
  • Communications networks
  • Transportation networks
EXERCISE
MST

• Show an MST of the following graph.
**Cycle Property**

- **Cycle Property:**
  - Let $T$ be a minimum spanning tree of a weighted graph $G$
  - Let $e$ be an edge of $G$ that is not in $T$ and $C$ let be the cycle formed by $e$ with $T$
  - For every edge $f$ of $C$, $\text{weight}(f) \leq \text{weight}(e)$

- **Proof by contradiction:**
  - If $\text{weight}(f) > \text{weight}(e)$ we can get a spanning tree of smaller weight by replacing $e$ with $f$
PARTITION PROPERTY

- **Partition Property:**
  - Consider a partition of the vertices of $G$ into subsets $U$ and $V$
  - Let $e$ be an edge of minimum weight across the partition
  - There is a minimum spanning tree of $G$ containing edge $e$

- **Proof by contradiction:**
  - Let $T$ be an MST of $G$
  - If $T$ does not contain $e$, consider the cycle $C$ formed by $e$ with $T$ and let $f$ be an edge of $C$ across the partition
  - By the cycle property, $\text{weight}(f) \leq \text{weight}(e)$
  - Thus, $\text{weight}(f) = \text{weight}(e)$
  - We obtain another MST by replacing $f$ with $e$
PRIM-JARNIK’S ALGORITHM

• We pick an arbitrary vertex $s$ and we grow the MST as a cloud of vertices, starting from $s$
• We store with each vertex $v$ a label $d(v)$ representing the smallest weight of an edge connecting $v$ to a vertex in the cloud
• At each step:
  • We add to the cloud the vertex $u$ outside the cloud with the smallest distance label
  • We update the labels of the vertices adjacent to $u$
PRIM-JARNIK’S ALGORITHM

- An adaptable priority queue stores the vertices outside the cloud
  - Key: distance, $D[v]$
  - Element: vertex $v$
  - $Q.replace(i, k)$ changes the key of an item
- We store three labels with each vertex $v$:
  - Distance $D[v]$
  - Parent edge in MST $P[v]$
  - Locator in priority queue

Algorithm PrimJarnikMST($G$)

Input: A weighted connected graph $G$

Output: A minimum spanning tree $T$ of $G$

1. Pick any vertex $s$ of $G$
2. $D[s] \leftarrow 0$; $P[s] \leftarrow \emptyset$
3. for each vertex $v \neq s$ do
4.   $D[v] \leftarrow \infty$; $P[v] \leftarrow \emptyset$
5.   $T \leftarrow \emptyset$
6.   Priority queue $Q$ of vertices with $D[v]$ as the key
7.   while $\neg Q.isEmpty()$ do
8.     $u \leftarrow Q.removeMin()$
9.     Add vertex $u$ and edge $P[u]$ to $T$
10.   for each $e \in u.outgoingEdges$ do
11.     $v \leftarrow G.\text{opposite}(u, e)$
12.     if $e.weight() < D[v]$ then
13.        $D[v] \leftarrow e.weight(); P[v] \leftarrow e$
14.        $Q.replace(v, D[v])$
15. return $T$
EXAMPLE
EXERCISE
PRIM’S MST ALGORITHM

• Show how Prim’s MST algorithm works on the following graph, assuming you start with SFO
  • Show how the MST evolves in each iteration (a separate figure for each iteration).
ANALYSIS

• Graph operations
  • Method incidentEdges is called once for each vertex

• Label operations
  • We set/get the distance, parent and locator labels of vertex \( z \) \( O(\deg(z)) \) times
  • Setting/getting a label takes \( O(1) \) time

• Priority queue operations
  • Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes \( O(\log n) \) time
  • The key of a vertex \( w \) in the priority queue is modified at most \( \deg(w) \) times, where each key change takes \( O(\log n) \) time

• Prim-Jarnik’s algorithm runs in \( O((n + m) \log n) \) time provided the graph is represented by the adjacency list structure
  • Recall that \( \Sigma_v \deg(v) = 2m \)
  • If the graph is connected the running time is \( O(m \log n) \)