CHAPTER 14
GRAPH ALGORITHMS

ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN JAVA, GOODRICH, TAMASSIA AND GOLDWASSER (WILEY 2016)
A **graph** is a pair $G = (V, E)$, where
- $V$ is a set of nodes, called **vertices**
- $E$ is a collection of pairs of vertices, called **edges**
- Vertices and edges can store arbitrary elements

**Example:**
- A vertex represents an airport and stores the three-letter airport code
- An edge represents a flight route between two airports and stores the mileage of the route
APPLICATIONS

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram
**TERMINOLOGY**

**EDGE AND GRAPH TYPES**

**Edge Types**
- **Directed** edge
  - ordered pair of vertices \((u, v)\)
  - first vertex \(u\) is the origin/source
  - second vertex \(v\) is the destination/target
  - e.g., a flight
- **Undirected** edge
  - unordered pair of vertices \((u, v)\)
  - e.g., a flight route
- **Weighted** edge
  - Numeric label associated with edge

**Graph Types**
- **Directed** graph (Digraph)
  - all the edges are directed
  - e.g., route network
- **Undirected** graph
  - all the edges are undirected
  - e.g., flight network
- **Weighted** graph
  - all the edges are weighted

\[(u, v)\]  
\[
\begin{array}{c}
\text{ORD} \quad \text{flight} \quad \text{DFW} \\
\text{AA 1206} \quad \text{802 miles} \\
\end{array}
\]
TERMINOLOGY
VERTICES AND EDGES

• **End points** (or end vertices) of an edge
  - $U$ and $V$ are the endpoints of $a$

• **Edges incident** on a vertex
  - $a$, $d$, and $b$ are incident on $V$

• **Adjacent** vertices
  - $U$ and $V$ are adjacent

• **Degree** of a vertex
  - $X$ has degree 5

• **Parallel (multiple)** edges
  - $h$ and $i$ are parallel edges

• **Self-loop**
  - $j$ is a self-loop

**Note:** A graph with no parallel edges or self loops are said to be simple. Unless otherwise stated, you should assume all graphs discussed are simple.
TERMINOLOGY
VERTICES AND EDGES

• **Outgoing** edges of a vertex
  • $h$ and $b$ are the outgoing edges of $X$

• **Incoming** edges of a vertex
  • $e$, $g$, and $i$ are incoming edges of $X$

• **In-degree** of a vertex
  • $X$ has in-degree 3

• **Out-degree** of a vertex
  • $X$ has out-degree 2
**TERMINOLOGY**

**PATHS**

- **Path**
  - Sequence of alternating vertices and edges
  - Begins with a vertex
  - Ends with a vertex
  - Each edge is preceded and followed by its endpoints

- **Simple path**
  - Path such that all its vertices and edges are distinct

- **Examples**
  - $P_1 = \{V, b, X, h, Z\}$ is a simple path
  - $P_2 = \{U, c, W, e, X, g, Y, f, W, d, V\}$ is a path that is not simple
TERMINOLOGY
Cycles

- **Cycle**
  - Circular sequence of alternating vertices and edges
  - Each edge is preceded and followed by its endpoints

- **Simple cycle**
  - Cycle such that all its vertices and edges are distinct except for the beginning and ending vertex

- **Examples**
  - $C_1 = \{V, b, X, Y, f, W, c, U, a, V\}$ is a simple cycle
  - $C_2 = \{U, c, W, e, X, g, Y, f, W, d, V, a, U\}$ is a cycle that is not simple

- A digraph is called **acyclic** if it does not contain any cycles
EXERCISE ON TERMINOLOGY

1. Number of vertices?
2. Number of edges?
3. What type of the graph is it?
4. Show the end vertices of the edge with largest weight
5. Show the vertices of smallest degree and largest degree
6. Show the edges incident to the vertices in the above question
7. Identify the shortest simple path from HNL to PVD
8. Identify the simple cycle with the most edges
EXERCISE
PROPERTIES OF UNDIRECTED GRAPHS

- Property 1 – Total degree
  \[ \sum_v \text{deg}(v) = ? \]

- Property 2 – Total number of edges
  - In an undirected graph with no self-loops and no multiple edges
    \[ m \leq \text{Upper Bound?} \]
    \[ \text{Lower Bound?} \leq m \]

- Notation
  - \( n \) number of vertices
  - \( m \) number of edges
  - \( \text{deg}(v) \) degree of vertex \( v \)

Example
- \( n = ? \)
- \( m = ? \)
- \( \text{deg}(v) = ? \)

A graph with given number of vertices (4) and maximum number of edges
EXERCISE
PROPERTIES OF UNDIRECTED GRAPHS

• Property 1 – Total degree
  \( \Sigma_v \text{deg}(v) = 2m \)

• Property 2 – Total number of edges
  • In an undirected graph with no self-loops and no multiple edges
    \[ m \leq \frac{n(n-1)}{2} \]
    \[ 0 \leq m \]
  Proof: Each vertex can have degree at most \( (n-1) \)

• Notation
  • \( n \) number of vertices
  • \( m \) number of edges
  • \( \text{deg}(v) \) degree of vertex \( v \)

Example
- \( n = 4 \)
- \( m = 6 \)
- \( \text{deg}(v) = 3 \)

A graph with given number of vertices (4) and maximum number of edges
EXERCISE
PROPERTIES OF DIRECTED GRAPHS

• Property 1 – Total in-degree and out-degree
  \[ \sum_{v} v_{in} - \deg(v) = ? \]
  \[ \sum_{v} v_{out} - \deg(v) = ? \]

• Property 2 – Total number of edges
  • In a directed graph with no self-loops and no multiple edges
    \[ m \leq \text{UpperBound?} \]
    \[ \text{LowerBound?} \leq m \]

• Notation
  • \( n \) number of vertices
  • \( m \) number of edges
  • \( \deg(v) \) degree of vertex \( v \)

Example
  • \( n =? \)
  • \( m =? \)
  • \( \deg(v) =? \)

A graph with given number of vertices (4) and maximum number of edges
EXERCISE
PROPERTIES OF DIRECTED GRAPHS

• Property 1 – Total in-degree and out-degree
  \[ \Sigma_{\text{in}} - \deg(v) = m \]
  \[ \Sigma_{\text{out}} - \deg(v) = m \]

• Property 2 – Total number of edges
  • In a directed graph with no self-loops and no multiple edges
    \[ m \leq n(n - 1) \]
    \[ 0 \leq m \]

• Notation
  • \( n \) number of vertices
  • \( m \) number of edges
  • \( \deg(v) \) degree of vertex \( v \)

Example
  • \( n = 4 \)
  • \( m = 12 \)
  • \( \deg(v) = 6 \)

A graph with given number of vertices (4) and maximum number of edges
TERMINOLOGY
CONNECTIVITY

• Given two vertices $u$ and $v$, we say $u$ reaches $v$, and that $v$ is reachable from $u$, if there exists a path from $u$ to $v$. In an undirected graph reachability is symmetric.

• A graph is connected if there is a path between every pair of vertices.

• A digraph is strongly connected if there every pair of vertices is reachable.
TERMINOLOGY

SUBGRAPHS

- A **subgraph** $H$ of a graph $G$ is a graph whose vertices and edges are subsets of $G$, respectively.
- A **spanning subgraph** of $G$ is a subgraph that contains all the vertices of $G$.
- A **connected component** of a graph $G$ is a maximal connected subgraph of $G$. 

Subgraph

Spanning subgraph

Non connected graph with two connected components
TERMINOLOGY
TREES AND FORESTS

• A **forest** is a graph without cycles

• A **(free) tree** is connected forest
  • This definition of tree is different from the one of a rooted tree

• The connected components of a forest are trees
SPANNING TREES AND FORESTS

• A **spanning tree** of a connected graph is a spanning subgraph that is a tree

• A spanning tree is not unique unless the graph is a tree

• Spanning trees have applications to the design of communication networks
GRAPH ADT

- Vertices and edges are lightweight objects and store elements.
- Although the ADT is specified from the graph object, we often have similar functions in the Vertex and Edge objects.

numVertices(): Returns the number of vertices of the graph.
vertices(): Returns an iteration of all the vertices of the graph.
numEdges(): Returns the number of edges of the graph.
edges(): Returns an iteration of all the edges of the graph.
getEdge(u, v): Returns the edge from vertex u to vertex v, if one exists; otherwise return null. For an undirected graph, there is no difference between getEdge(u, v) and getEdge(v, u).
endVertices(e): Returns an array containing the two endpoint vertices of edge e. If the graph is directed, the first vertex is the origin and the second is the destination.
opposite(v, e): For edge e incident to vertex v, returns the other vertex of the edge; an error occurs if e is not incident to v.
outDegree(v): Returns the number of outgoing edges from vertex v.
inDegree(v): Returns the number of incoming edges to vertex v. For an undirected graph, this returns the same value as does outDegree(v).
outgoingEdges(v): Returns an iteration of all outgoing edges from vertex v.
incomingEdges(v): Returns an iteration of all incoming edges to vertex v. For an undirected graph, this returns the same collection as does outgoingEdges(v).
insertVertex(x): Creates and returns a new Vertex storing element x.
insertEdge(u, v, x): Creates and returns a new Edge from vertex u to vertex v, storing element x; an error occurs if there already exists an edge from u to v.
removeVertex(v): Removes vertex v and all its incident edges from the graph.
removeEdge(e): Removes edge e from the graph.
EXERCISE ON ADT

1. `outgoingEdges(ord)`
2. `incomingEdges(ord)`
3. `outDegree(ord)`
4. `endVertices({lga, mia})`
5. `opposite(dfw, {dfw, lga})`

6. `insertVertex(iah)`
7. `insertEdge(mia, pvd, 1200)`
8. `removeVertex(ord)`
9. `removeEdge({dfw, ord})`
An edge list can be stored in a list or a map/dictionary (e.g. hash table).

- Vertex object:
  - element
  - reference to position in vertex sequence

- Edge object:
  - element
  - origin vertex object
  - destination vertex object
  - reference to position in edge sequence
EXERCISE
EDGE LIST STRUCTURE

• Construct the edge list for the following graph
ASYMPTOTIC PERFORMANCE
EDGE LIST STRUCTURE

- $n$ vertices, $m$ edges
- No parallel edges
- No self-loops

<table>
<thead>
<tr>
<th>Space</th>
<th>Edge List</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

getEdge($u, v$), outDegree($v$), outgoingEdges($v$), insertEdge($u, v, w$), removeVertex($v$)

insertVertex($x$), removeEdge($e$)
## Asymptotic Performance

### Edge List Structure

- **n** vertices, **m** edges
- No parallel edges
- No self-loops

<table>
<thead>
<tr>
<th>Space</th>
<th>Edge List</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(n + m)$</td>
<td><strong>Edge List</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Action</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>getEdge(u, v)</code></td>
<td>$O(m)$</td>
</tr>
<tr>
<td><code>outDegree(v)</code></td>
<td>$O(m)$</td>
</tr>
<tr>
<td><code>outgoingEdges(v)</code></td>
<td>$O(m)$</td>
</tr>
<tr>
<td><code>insertEdge(u, v, w)</code></td>
<td>$O(m)$</td>
</tr>
<tr>
<td><code>removeVertex(v)</code></td>
<td>$O(m)$</td>
</tr>
<tr>
<td><code>insertVertex(x)</code></td>
<td>$O(n + m)$</td>
</tr>
<tr>
<td><code>removeEdge(e)</code></td>
<td>$O(n + m)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Edge List</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><code>{ORD, PVD, 849}</code></td>
</tr>
<tr>
<td><code>{ORD, DFW, 802}</code></td>
</tr>
<tr>
<td><code>{LGA, PVD, 142}</code></td>
</tr>
<tr>
<td><code>{LGA, MIA, 1099}</code></td>
</tr>
<tr>
<td><code>{DFW, LGA, 1387}</code></td>
</tr>
<tr>
<td><code>{DFW, MIA, 1120}</code></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Vertex List</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>ORD</td>
</tr>
<tr>
<td>LGA</td>
</tr>
<tr>
<td>PVD</td>
</tr>
<tr>
<td>DFW</td>
</tr>
<tr>
<td>MIA</td>
</tr>
</tbody>
</table>
ADJACENCY LIST STRUCTURE

• **Adjacency Lists** associate vertices with their edges (in addition to edge list!)

• Each vertex stores a list of incident edges
  • List of references to incident edge objects

• Augmented edge object
  • Stores references to associated positions in incident adjacency lists

**Adjacency List**

- **ORD**
  - {ORD, PVD}
  - {ORD, DFW}

- **LGA**
  - {LGA, PVD}
  - {LGA, MIA}
  - {LGA, DFW}

- **PVD**
  - {PVD, ORD}
  - {PVD, LGA}

- **DFW**
  - {DFW, ORD}
  - {DFW, LGA}
  - {DFW, MIA}

- **MIA**
  - {MIA, LGA}
  - {MIA, DFW}
EXERCISE
ADJACENCY LIST STRUCTURE

• Construct the adjacency list for the following graph.
**ASYMPTOTIC PERFORMANCE**

**ADJACENCY LIST STRUCTURE**

<table>
<thead>
<tr>
<th>Space</th>
<th>Adjacency List</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ vertices, $m$ edges</td>
<td></td>
</tr>
<tr>
<td>No parallel edges</td>
<td>Adjacency List</td>
</tr>
<tr>
<td>No self-loops</td>
<td></td>
</tr>
<tr>
<td>getEdge($u, v$), insertEdge($u, v, w$)</td>
<td></td>
</tr>
<tr>
<td>outDegree($v$), insertVertex($x$), removeEdge($e$)</td>
<td></td>
</tr>
<tr>
<td>outgoingEdges($v$), removeVertex($v$)</td>
<td></td>
</tr>
</tbody>
</table>

Adjacency List:
- **ORD**: {ORD, PVD} {ORD, DFW}
- **LGA**: {LGA, PVD} {LGA, MIA} {LGA, DFW}
- **PVD**: {PVD, ORD} {PVD, LGA}
- **DFW**: {DFW, ORD} {DFW, LGA} {DFW, MIA}
- **MIA**: {MIA, LGA} {MIA, DFW}
### ASYMPTOTIC PERFORMANCE

**ADJACENCY LIST STRUCTURE**

- $n$ vertices, $m$ edges
- No parallel edges
- No self-loops

<table>
<thead>
<tr>
<th>Operation</th>
<th>Adjacency List</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Space</strong></td>
<td>$O(n + m)$</td>
</tr>
<tr>
<td>getEdge($u, v$), insertEdge($u, v, w$)</td>
<td>$O(\min(\text{deg}(v), \text{deg}(u)))$</td>
</tr>
<tr>
<td>outDegree($v$), insertVertex($x$), removeEdge($e$)</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>outgoingEdges($v$), removeVertex($v$)</td>
<td>$O(\text{deg}(v))$</td>
</tr>
</tbody>
</table>

**Adjacency List**

- ORD: {ORD, PVD} → {ORD, DFW}
- LGA: {LGA, PVD} → {LGA, MIA} → {LGA, DFW}
- PVD: {PVD, ORD} → {PVD, LGA}
- DFW: {DFW, ORD} → {DFW, LGA} → {DFW, MIA}
- MIA: {MIA, LGA} → {MIA, DFW}
• We can store augmenting incidence structures in maps, instead of lists. This is called an **adjacency map**.
  - In general an "adjacency list" means storing adjacency with vertices, so the terms are interchangeable

• What would this do to the complexities?
  - If it is implemented as a hash table?
  - If it is implemented as a red-black tree?
### Adjacency Matrix Structure

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>∅</td>
<td>∅</td>
<td>{0, 2}</td>
<td>{0, 3}</td>
<td>∅</td>
</tr>
<tr>
<td>1</td>
<td>∅</td>
<td>∅</td>
<td>{1, 2}</td>
<td>{1, 3}</td>
<td>{1, 4}</td>
</tr>
<tr>
<td>2</td>
<td>{0, 2}</td>
<td>{1, 2}</td>
<td>∅</td>
<td>∅</td>
<td>∅</td>
</tr>
<tr>
<td>3</td>
<td>{0, 3}</td>
<td>{1, 3}</td>
<td>∅</td>
<td>∅</td>
<td>{3, 4}</td>
</tr>
<tr>
<td>4</td>
<td>∅</td>
<td>{1, 4}</td>
<td>∅</td>
<td>{3, 4}</td>
<td>∅</td>
</tr>
</tbody>
</table>

- Adjacency matrices store references to edges in a table (in addition to the edge list)
- Augment vertices with integer keys (often done in all graph implementations!)

---

```
0: ORD
2: PVD
3: DFW
4: MIA
1: LGA
```
EXERCISE

ADJACENCY MATRIX STRUCTURE

• Construct the adjacency matrix for the following graph.
ASYMPTOTIC PERFORMANCE

ADJACENCY MATRIX STRUCTURE

<table>
<thead>
<tr>
<th>Space</th>
<th>Adjacency Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>• $n$ vertices, $m$ edges</td>
<td></td>
</tr>
<tr>
<td>• No parallel edges</td>
<td></td>
</tr>
<tr>
<td>• No self-loops</td>
<td></td>
</tr>
</tbody>
</table>

```
0 1 2 3 4
0 0 0 {0, 2} {0, 3} 0
1 0 0 {1, 2} {1, 3} {1, 4}
2 {0, 2} {1, 2} 0 0 0
3 {0, 3} {1, 3} 0 0 {3, 4}
4 0 {1, 4} 0 {3, 4} 0
```

- `outDegree(v)`, `outgoingEdges(v)`
- `getEdge(u, v)`, `insertEdge(u, v, w)`, `removeEdge(e)`
- `insertVertex(x)`, `removeVertex(v)`
### Asymptotic Performance

**Adjacency Matrix Structure**

- **$n$** vertices, **$m$** edges
- No parallel edges
- No self-loops

<table>
<thead>
<tr>
<th>Space</th>
<th>Adjacency Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td><code>outDegree(v)</code></td>
<td>$O(n)$</td>
</tr>
<tr>
<td><code>outgoingEdges(v)</code></td>
<td></td>
</tr>
<tr>
<td><code>getEdge(u, v)</code></td>
<td></td>
</tr>
<tr>
<td><code>insertEdge(u, v, w)</code></td>
<td>$O(1)$</td>
</tr>
<tr>
<td><code>removeEdge(e)</code></td>
<td></td>
</tr>
<tr>
<td><code>insertVertex(x)</code></td>
<td></td>
</tr>
<tr>
<td><code>removeVertex(v)</code></td>
<td>$O(n^2)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>{0, 2}</td>
<td>{0, 3}</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>1</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>{1, 2}</td>
<td>{1, 3}</td>
<td>{1, 4}</td>
</tr>
<tr>
<td>2</td>
<td>{0, 2}</td>
<td>{1, 2}</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>3</td>
<td>{0, 3}</td>
<td>{1, 3}</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>{3, 4}</td>
</tr>
<tr>
<td>4</td>
<td>$\emptyset$</td>
<td>{1, 4}</td>
<td>$\emptyset$</td>
<td>{3, 4}</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>
## ASYMPTOTIC PERFORMANCE

- $n$ vertices, $m$ edges
- No parallel edges
- No self-loops

<table>
<thead>
<tr>
<th></th>
<th>Edge List</th>
<th>Adjacency List</th>
<th>Adjacency Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Space</strong></td>
<td>$O(n + m)$</td>
<td>$O(n + m)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td><strong>outgoingEdges(v)</strong></td>
<td>$O(m)$</td>
<td>$O(\text{deg}(v))$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td><strong>getEdge(u, v)</strong></td>
<td>$O(m)$</td>
<td>$O(\min(\text{deg}(v), \text{deg}(w)))$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td><strong>insertEdge(u, v, w)</strong></td>
<td>$O(m)$</td>
<td>$O(\min(\text{deg}(v), \text{deg}(w)))$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td><strong>eraseEdge(e)</strong></td>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td><strong>insertVertex(x)</strong></td>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td><strong>removeVertex(v)</strong></td>
<td>$O(m)$</td>
<td>$O(\text{deg}(v))$</td>
<td>$O(n^2)$</td>
</tr>
</tbody>
</table>