CH9.
PRIORITY QUEUES

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PRIORITY QUEUES

• Stores a collection of elements each with an associated “key” value
  • Can insert as many elements in any order
  • Only can inspect and remove a single element – the minimum (or maximum depending) element

• Applications
  • Standby Flyers
  • Auctions
  • Stock market
PRIORITY QUEUE ADT

- A priority queue stores a collection of entries
- Each entry is a pair (key, value)
- Main methods of the Priority Queue ADT
  - Entry `insert(k, v)` inserts an entry with key k and value v
  - Entry `removeMin()` removes and returns the entry with smallest key, or null if the priority queue is empty
- Additional methods
  - Entry `min()` returns, but does not remove, an entry with smallest key, or null if the priority queue is empty
  - `size()`, `isEmpty()`
TOTAL ORDER RELATION

- Keys in a priority queue can be arbitrary objects on which an order is defined, e.g., integers
- Two distinct items in a priority queue can have the same key

- Mathematical concept of total order relation ≤
  - Reflexive property: \( k \leq k \)
  - Antisymmetric property: if \( k_1 \leq k_2 \) and \( k_2 \leq k_1 \), then \( k_1 = k_2 \)
  - Transitive property: if \( k_1 \leq k_2 \) and \( k_2 \leq k_3 \) then \( k_1 \leq k_3 \)
ENTRY ADT

• An entry in a priority queue is simply a key-value pair

• Priority queues store entries to allow for efficient insertion and removal based on keys

• Methods:
  • Key `getKey()`: returns the key for this entry
  • Value `getValue()`: returns the value associated with this entry
COMPARATOR ADT

• A comparator encapsulates the action of comparing two objects according to a given total order relation

• A generic priority queue uses an auxiliary comparator, i.e., it is external to the keys being compared

• When the priority queue needs to compare two keys, it uses its comparator

• Primary method of the Comparator ADT

  \textbf{Integer \texttt{compare}(x, y): returns an integer } i \textbf{ such that}

  • \( i < 0 \) if \( x < y \),
  • \( i = 0 \) if \( x = y \)
  • \( i > 0 \) if \( x > y \)
  • An error occurs if a and b cannot be compared.
We can use a priority queue to sort a set of comparable elements.

Insert the elements one by one with a series of `insert(e)` operations.

Remove the elements in sorted order with a series of `removeMin()` operations.

Running time depends on the PQ implementation.

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**Algorithm** PriorityQueueSort()

**Input:** List $L$ storing $n$ elements and a Comparator $C$

**Output:** Sorted List $L$

1. Priority Queue $P$ using comparator $C$

2. while $¬L.isEmpty()$ do

3. $P.insert(L.first())$

4. $L.removeFirst()$

5. while $¬P.isEmpty()$ do

6. $L.insertLast(P.min())$

7. $P.removeMin()$

8. *return* $L$
LIST-BASED PRIORITY QUEUE

Unsorted list implementation

- Store the items of the priority queue in a list, in arbitrary order

• Performance:
  - `insert(e)` takes $O(1)$ time since we can insert the item at the beginning or end of the list
  - `removeMin()` and `min()` take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

Sorted list implementation

- Store the items of the priority queue in a list, sorted by key

• Performance:
  - `insert(e)` takes $O(n)$ time since we have to find the place where to insert the item
  - `removeMin()` and `min()` take $O(1)$ time since the smallest key is at the beginning of the list
Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted list.

Running time of Selection-sort:
- Inserting the elements into the priority queue with \( n \) \texttt{insert(e)} operations takes \( O(n) \) time.
- Removing the elements in sorted order from the priority queue with \( n \) \texttt{removeMin()} operations takes time proportional to

\[
\sum_{i=0}^{n} n - i = n + (n - 1) + \cdots + 2 + 1 = O(n^2)
\]

Selection-sort runs in \( O(n^2) \) time.
EXERCISE
SELECTION-SORT

• Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted list (do $n$ insert($e$) and then $n$ removeMin())

• Illustrate the performance of selection-sort on the following input sequence:
  • (22, 15, 36, 44, 10, 3, 9, 13, 29, 25)
• Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted List

• Running time of Insertion-sort:
  • Inserting the elements into the priority queue with $n \text{ insert}(e)$ operations takes time proportional to
    $$\sum_{i=0}^{n} i = 1 + 2 + \cdots + n = O(n^2)$$
  • Removing the elements in sorted order from the priority queue with a series of $n \text{ removeMin}()$ operations takes $O(n)$ time

• Insertion-sort runs in $O(n^2)$ time
EXERCISE
INSERTION-SORT

• Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted list (do $n \text{ insert}(e)$ and then $n \text{ removeMin()}$)

• Illustrate the performance of insertion-sort on the following input sequence:
  • (22, 15, 36, 44, 10, 3, 9, 13, 29, 25)
In-Place Insertion-Sort

- Instead of using an external data structure, we can implement selection-sort and insertion-sort in-place (only $O(1)$ extra storage)
- A portion of the input list itself serves as the priority queue
- For in-place insertion-sort
  - We keep sorted the initial portion of the list
  - We can use $\text{swap}(i, j)$ instead of modifying the list
HEAPS
WHAT IS A HEAP?

• A heap is a binary tree storing keys at its internal nodes and satisfying the following properties:
  • **Heap-Order**: for every node \( v \) other than the root, \( \text{key}(v) \geq \text{key}(v.\text{parent})() \)
  • **Complete Binary Tree**: let \( h \) be the height of the heap
    • for \( i = 0 \ldots h - 1 \), there are \( 2^i \) nodes on level \( i \)
    • at level \( h - 1 \), nodes are filled from left to right

• Can be used to store a priority queue efficiently

```
  2
  / \   \\
 5   6
 /   /   \\
9   7   \\
```

last node
HEIGHT OF A HEAP

- **Theorem:** A heap storing \( n \) keys has height \( O(\log n) \)
- **Proof:** (we apply the complete binary tree property)
  - Let \( h \) be the height of a heap storing \( h \) keys
  - Since there are \( 2^i \) keys at level \( i = 0 \) ... \( h - 1 \) and at least one key on level \( h \), we have
    \[
    n \geq 1 + 2 + 4 + \cdots + 2^{h-1} + 1 = (2^h - 1) + 1 = 2^h
    \]
  - Level \( h \) has at most \( 2^h \) nodes: \( n \leq 2^{h+1} - 1 \)
  - Thus, \( \log(n + 1) - 1 \leq h \leq \log n \)

<table>
<thead>
<tr>
<th>depth</th>
<th>keys</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( h - 1 )</td>
<td>( 2^{h-1} )</td>
</tr>
<tr>
<td>( h )</td>
<td>1</td>
</tr>
</tbody>
</table>
EXERCISE
HEAPS

• Let $H$ be a heap with 7 distinct elements (1, 2, 3, 4, 5, 6, and 7). Is it possible that a preorder traversal visits the elements in sorted order? What about an inorder traversal or a postorder traversal? In each case, either show such a heap or prove that none exists.
INSERTION INTO A HEAP

• **insert(e)** consists of three steps
  • Find the insertion node \( z \) (the new last node)
  • Store \( e \) at \( z \) and expand \( z \) into an internal node
  • Restore the heap-order property (discussed next)
UPHEAP

- After the insertion of a new element \( e \), the heap-order property may be violated.
- Up-heap bubbling restores the heap-order property by swapping \( e \) along an upward path from the insertion node.
- Upheap terminates when \( e \) reaches the root or a node whose parent has a key smaller than or equal to \( \text{key}(e) \).
- Since a heap has height \( O(\log n) \), upheap runs in \( O(\log n) \) time.
REMOVAL FROM A HEAP

- \texttt{removeMin()} corresponds to the removal of the root from the heap

- The removal algorithm consists of three steps
  - Replace the root with the element of the last node \( w \)
  - Compress \( w \) and its children into a leaf
  - Restore the heap-order property (discussed next)
DOWNHEAP

- After replacing the root element of the last node, the heap-order property may be violated.
- **Down-heap bubbling** restores the heap-order property by swapping element $e$ along a downward path from the root.
- Downheap terminates when $e$ reaches a leaf or a node whose children have keys greater than or equal to $\text{key}(e)$.
- Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time.
UPDATING THE LAST NODE

• The insertion node can be found by traversing a path of $O(\log n)$ nodes
  • Go up until a left child or the root is reached
  • If a left child is reached, go to the right child
  • Go down left until a leaf is reached

• Similar algorithm for updating the last node after a removal
HEAP-SORT

- Consider a priority queue with $n$ items implemented by means of a heap
  - the space used is $O(n)$
  - `insert(e)` and `removeMin()` take $O(\log n)$ time
  - `min()`, `size()`, and `empty()` take $O(1)$ time

- Using a heap-based priority queue, we can sort a sequence of $n$ elements in $O(n \log n)$ time
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort
EXERCISE
HEAP-SORT

• Heap-sort is the variation of PQ-sort where the priority queue is implemented with a heap (do \texttt{n insert(e)} and then \texttt{n removeMin()})

• Illustrate the performance of heap-sort on the following input sequence (draw the heap at each step):
  • \((22, 15, 36, 44, 10, 3, 9, 13, 29, 25)\)
ARRAY-BASED HEAP IMPLEMENTATION

• We can represent a heap with \( n \) elements by means of a vector of length \( n \)
  • Links between nodes are not explicitly stored
  • The leaves are not represented
  • The cell at index 0 is the root

• For the node at index \( i \)
  • the left child is at index \( 2i + 1 \)
  • the right child is at index \( 2i + 2 \)

• \texttt{insert(e)} corresponds to inserting at index \( n + 1 \)

• \texttt{removeMin()} corresponds to removing element at index \( n \)

• Yields in-place heap-sort

![Heap Diagram and Array Representation](image-url)
## Priority Queue Summary

<table>
<thead>
<tr>
<th></th>
<th>insert(e)</th>
<th>removeMin()</th>
<th>PQ-Sort total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordered List</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>(Insertion Sort)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unordered List</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>(Selection Sort)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Binary Heap,</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
<td>$O(n \log n)$</td>
</tr>
<tr>
<td>Vector-based Heap</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Heap Sort)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
MERGING TWO HEAPS

• We are given two heaps and a new element $e$
• We create a new heap with a root node storing $e$ and with the two heaps as subtrees
• We perform downheap to restore the heap-order property
We can construct a heap storing $n$ given elements in using a bottom-up construction with $\log n$ phases.

In phase $i$, pairs of heaps with $2^i - 1$ elements are merged into heaps with $2^{i+1} - 1$ elements.
EXAMPLE
EXAMPLE
ANALYSIS

• We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path).

• Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$.

• Thus, bottom-up heap construction runs in $O(n)$ time.

• Bottom-up heap construction is faster than $n$ successive insertions and speeds up the first phase of heap-sort.
One weakness of the priority queues so far is that we do not have an ability to update individual entries, like in a changing price market or bidding service.

Recall that \( \text{insert}(e) \) returns an entry. We need to save these values to be able to adapt them.

Additional ADT support (also includes standard priority queue functionality):

- **Entry \( \text{remove}(e) \) — remove a specific entry \( e \)**
- **Key \( \text{replaceKey}(e, k) \) — replace the key of entry \( e \) with \( k \), and return the old key.**
- **Value \( \text{replaceValue}(e, k) \) — replace the value of entry \( e \) with \( k \), and return the old value.**
• **Locators** decouple positions and entries in order to support efficient adaptable priority queue implementations (i.e., in a heap)

• Each position has an associated locator

• Each locator stores a pointer to its position and memory for the entry
POSITIONS VS. LOCATORS

- **Position**
  - represents a “place” in a data structure
  - related to other positions in the data structure (e.g., previous/next or parent/child)
  - often implemented as a pointer to a node or the index of an array cell

- **Locator**
  - identifies and tracks a (key, element) item
  - unrelated to other locators in the data structure
  - often implemented as an object storing the item and its position in the underlying structure

- **Position-based ADTs** (e.g., sequence and tree) are fundamental data storage schemes

- **Key-based ADTs** (e.g., priority queue) can be augmented with locator-based methods