Solutions to the Sixtieth William Lowell Putnam Mathematical Competition
Saturday, December 4, 1999
Prepared by Manjul Bhargava, Kiran Kedlaya, and Lenny Ng

A–1 Note that if \( r(x) \) and \( s(x) \) are any two functions, then
\[
\max(r, s) = \frac{(r + s + |r - s|)}{2}.
\]
Therefore, if \( F(x) \) is the given function, we have
\[
F(x) = \max\{-3x - 3, 0\} - \max\{5x, 0\} + 3x + 2
\]
\[
= (-3x - 3 + |3x - 3|)/2 - (5x + |5x|)/2 + 3x + 2
\]
\[
= |(3x - 3)/2| - |5x/2| - x + \frac{1}{2},
\]
so we may set \( f(x) = (3x - 3)/2, \ g(x) = 5x/2, \) and \( h(x) = -x + \frac{1}{2} \).

A–2 First factor \( p(x) = q(x)r(x) \), where \( q \) has all real roots and \( r \) has all complex roots. Notice that each root of \( q \) has even multiplicity, otherwise \( p \) would have a sign change at that root. Thus \( q(x) \) has a square root \( s(x) \).

Now write \( r(x) = \prod_{j=1}^{b} (x - a_j)(x - \overline{a_j}) \) (possible because \( r \) has roots in complex conjugate pairs). Write \( \prod_{j=1}^{b} (x - a_j) = t(x) + iu(x) \) with \( t, x \) having real coefficients. Then for \( x \) real,
\[
p(x) = q(x)r(x) = s(x)^2(t(x) + iu(x))(\overline{t(x)} + iu(x)) = (s(x)t(x))^2 + (s(x)u(x))^2.
\]

A–3 First solution: Computing the coefficient of \( x^{n+1} \) in the identity \( (1 - 2x - x^2) \sum_{m=0}^{\infty} a_m x^m = 1 \) yields the recurrence \( a_{n+1} = 2a_n + a_{n-1} \); the sequence \( \{a_n\} \) is then characterized by this recurrence and the initial conditions \( a_0 = 1, a_1 = 2 \).

Define the sequence \( \{b_n\} \) by \( b_{2n} = a_{n-1}^2 + a_n^2, \ b_{2n+1} = a_n(a_{n-1} + a_{n+1}) \). Then
\[
2b_{2n+1} + b_{2n} = 2a_n a_{n+1} + 2a_{n-1} a_n + a_{n-1}^2 + a_n^2
\]
\[
= 2a_n a_{n+1} + a_{n-1} a_{n+1} + a_n^2
\]
\[
= a_n^2 + a_{n+1}^2 = b_{2n+2},
\]
and similarly \( 2b_{2n} + b_{2n-1} = b_{2n+1} \), so that \( \{b_n\} \) satisfies the same recurrence as \( \{a_n\} \).

Since further \( b_0 = 1, b_1 = 2 \) (where we use the recurrence for \( \{a_n\} \) to calculate \( a_{-1} = 0 \)), we deduce that \( b_n = a_n \) for all \( n \). In particular, \( a_n^2 + a_{n+1}^2 = b_{2n+2} = a_{2n+2} \).

Second solution: Note that
\[
\frac{1}{1 - 2x - x^2} = \frac{1}{2\sqrt{2}} \left( \frac{\sqrt{2} + 1}{1 - (\sqrt{2} + 1)x} + \frac{\sqrt{2} - 1}{1 - (\sqrt{2} - 1)x} \right)
\]
\[
= \frac{1}{2\sqrt{2}} \left( \sum_{n=0}^{\infty} (\sqrt{2} + 1)^n x^n - \sum_{n=0}^{\infty} (\sqrt{2} - 1)^{n+1} x^n \right),
\]
so that \( a_n = \frac{1}{2\sqrt{2}} \left( (\sqrt{2} + 1)^{n+1} - (1 - \sqrt{2})^{n+1} \right) \). A simple computation (omitted here) now shows that \( a_n^2 + a_{n+1}^2 = a_{2n+2} \).

A–4 Denote the series by \( S \), and note that

\[
S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(3^m/m)(3^m/m + 3^n/n)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(3^n/n)(3^n/m + 3^n/n)},
\]

where the second equality follows by interchanging \( m \) and \( n \). Thus

\[
2S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{1}{(3^m/m)(3^m/m + 3^n/n)} + \frac{1}{(3^n/n)(3^n/m + 3^n/n)} \right)
\]

\[
= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(3^m/m)(3^n/n)}
\]

\[
= \left( \sum_{n=1}^{\infty} \frac{n}{3^n} \right)^2.
\]

But \( \sum_{n=1}^{\infty} n/3^n = 3/4 \) (since, e.g., it’s \( f'(1) \), where \( f(x) = \sum_{n=0}^{\infty} x^n/3^n = 3/(3 - x) \)), and we conclude that \( S = 9/32 \).

A–5 First solution: (by Reid Barton) Let \( r_1, \ldots, r_{1999} \) be the roots of \( P \). Draw a disc of radius \( \epsilon \) around each \( r_i \), where \( \epsilon < 1/3998 \); this disc covers a subinterval of \([-1/2, 1/2]\) of length at most \( 2\epsilon \), and so of the 2000 (or fewer) uncovered intervals in \([-1/2, 1/2]\), one, which we call \( I \), has length at least \( \delta = (1 - 3998\epsilon)/2000 > 0 \). We will exhibit an explicit lower bound for the integral of \( |P(x)|/P(0) \) over this interval, which will yield such a bound for the entire integral.

Note that

\[
\frac{|P(x)|}{|P(0)|} = \prod_{i=1}^{1999} \frac{|x - r_i|}{|r_i|}.
\]

Also note that by construction, \( |x - r_i| \geq \epsilon \) for each \( x \in I \). If \( |r_i| \leq 1 \), then we have \( \frac{|x - r_i|}{|r_i|} \geq \epsilon \). If \( |r_i| > 1 \), then

\[
\frac{|x - r_i|}{|r_i|} = |1 - x/r_i| \geq 1 - |x/r_i| \geq 1 - 1/2 = 1/2 > \epsilon.
\]

We conclude that \( \int_I |P(x)/P(0)| \, dx \geq \delta \epsilon \), independent of \( P \).

Second solution: It will be a bit more convenient to assume \( P(0) = 1 \) (which we may achieve by rescaling unless \( P(0) = 0 \), in which case there is nothing to prove) and to prove that there exists \( D > 0 \) such that \( \int_{-1}^{1} |P(x)| \, dx \geq D \), or even such that \( \int_{0}^{1} |P(x)| \, dx \geq D \).

We first reduce to the case where \( P \) has all of its roots in \([0, 1]\). If this is not the case, we can factor \( P(x) \) as \( Q(x)R(x) \), where \( Q \) has all roots in the interval and \( R \) has none.
Then $R$ is either always positive or always negative on $[0, 1]$; assume the former. Let $k$ be the largest positive real number such that $R(x) - kx \geq 0$ on $[0, 1]$; then
\[
\int_{-1}^{1} |P(x)|\,dx = \int_{-1}^{1} |Q(x)R(x)|\,dx > \int_{-1}^{1} |Q(x)(R(x) - kx)|\,dx,
\]
and $Q(x)(R(x) - kx)$ has more roots in $[0, 1]$ than does $P$ (and has the same value at 0). Repeating this argument shows that $\int_{0}^{1} |P(x)|\,dx$ is greater than the corresponding integral for some polynomial with all of its roots in $[0, 1]$.

Under this assumption, we have $P(x) = c \prod_{i=1}^{1999} (x - r_i)$ for some $r_i \in (0, 1]$. Since $P(0) = -c \prod r_i = 1$, we have $|c| \geq \prod |r_i^{-1}| \geq 1$.

Thus it suffices to prove that if $Q(x)$ is a monic polynomial of degree 1999 with all of its roots in $[0, 1]$, then $\int_{0}^{1} |Q(x)|\,dx \geq D$ for some constant $D > 0$. But the integral of $\int_{0}^{1} \prod_{i=1}^{1999} |x - r_i|\,dx$ is a continuous function for $r_i \in [0, 1]$. The product of all of these intervals is compact, so the integral achieves a minimum value for some $r_i$. This minimum is the desired $D$.

Note: combining the two approaches gives a constructive solution with a constant that is better, but is still far from optimal. I don’t know offhand whether it is even known what the optimal constant and/or the polynomials achieving that constant are.

A–6 Rearranging the given equation yields the much more tractable equation
\[
\frac{a_n}{a_{n-1}} = 6 \frac{a_{n-1}}{a_{n-2}} - 8 \frac{a_{n-2}}{a_{n-3}}.
\]
Let $b_n = a_n/a_{n-1}$; with the initial conditions $b_2 = 2, b_3 = 12$, one easily obtains $b_n = 2^{n-1}(2^{n-2} - 1)$, and so
\[
a_n = 2^{n(n-1)/2} \prod_{i=1}^{n-1} (2^i - 1).
\]
To see that $n$ divides $a_n$, factor $n$ as $2^k m$, with $m$ odd. Then note that $k \leq n \leq n(n-1)/2$, and that there exists $i \leq m - 1$ such that $m$ divides $2^i - 1$, namely $i = \phi(m)$ (Euler’s totient function: the number of integers in $\{1, \ldots, m\}$ relatively prime to $m$).

B–1 The answer is 1/3. Let $G$ be the point obtained by reflecting $C$ about the line $AB$.
Since $\angle ADC = \frac{\pi - \theta}{2}$, we find that $\angle BDE = \pi - \theta - \angle ADC = \frac{\pi - \theta}{2} = \angle ADC = \pi - \angle BDC = \pi - \angle BDG$, so that $E, D, G$ are collinear. Hence
\[
|EF| = \frac{|BE|}{|BC|} = \frac{|BE|}{|BG|} = \frac{\sin(\theta/2)}{\sin(3\theta/2)},
\]
where we have used the law of sines in $\triangle BDG$. But by l’Hôpital’s Rule, $\lim_{\theta \to 0} \frac{\sin(\theta/2)}{\sin(3\theta/2)} = \lim_{\theta \to 0} \frac{\cos(\theta/2)}{3\cos(3\theta/2)} = 1/3$.  

3
B–2 Suppose that $P$ does not have $n$ distinct roots; then it has a root of multiplicity at least 2, which we may assume is $x = 0$ without loss of generality. Let $x^k$ be the greatest power of $x$ dividing $P(x)$, so that $P(x) = x^k R(x)$ with $R(0) \neq 0$; a simple computation yields

$$P''(x) = k(k-1)x^{k-2}R(x) + 2kx^{k-1}R'(x) + x^k R''(x).$$

Since $R(0) \neq 0$ and $k \geq 2$, we conclude that the greatest power of $x$ dividing $P''(x)$ is $x^{k-2}$. But $P(x) = Q(x)P''(x)$, and so $x^2$ divides $Q(x)$. We deduce (since $Q$ is quadratic) that $Q(x)$ is a constant $C$ times $x^2$; in fact, $C = 1/(n(n-1))$ by inspection of the leading-degree terms of $P(x)$ and $P''(x)$.

Now if $P(x) = \sum_{j=0}^{n} a_j x^j$, then the relation $P(x) = Cx^2 P''(x)$ implies that $a_j = Cj(j-1)a_j$ for all $j$; hence $a_j = 0$ for $j \leq n-1$, and we conclude that $P(x) = a_n x^n$, which has all identical roots.

B–3 We first note that

$$\sum_{m,n>0} x^m y^n = \frac{xy}{(1-x)(1-y)}.$$

Subtracting $S$ from this gives two sums, one of which is

$$\sum_{m \geq 2n+1} x^m y^n = \sum_{n} y^n \frac{x^{2n+1}}{1-x} = \frac{x^3 y}{1-x(1-x^2 y)}$$

and the other of which sums to $xy^3/[(1-y)(1-xy^2)]$. Therefore

$$S(x,y) = \frac{xy}{(1-x)(1-y)} - \frac{x^3 y}{(1-x)(1-x^2 y)} - \frac{xy^3}{(1-y)(1-xy^2)}$$

and the desired limit is $\lim_{(x,y) \to (1,1)} xy(1 + x + y + xy - x^2 y^2) = 3$.

B–4 We make repeated use of the following fact: if $f$ is a differentiable function on all of $\mathbb{R}$, $\lim_{x \to -\infty} f(x) \geq 0$, and $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$.

(Proof: if $f(y) < 0$ for some $x$, then $f(x) < f(y)$ for all $x < y$ since $f' > 0$, but then $\lim_{x \to -\infty} f(x) = f(y) < 0$.)

From the inequality $f'''(x) \leq f(x)$ we obtain

$$f'' f'''(x) \leq f''(x) f(x) < f''(x) f(x) + f'(x)^2$$

since $f'(x)$ is positive. Applying the fact to the difference between the right and left sides, we get

$$\frac{1}{2} (f''(x))^2 < f(x) f'(x).$$
Adding $\frac{1}{2}f'(x)f'''(x)$ to both sides and again invoking the original bound $f'''(x) \leq f(x)$, we get

$$\frac{1}{2}[f'(x)f'''(x) + (f''(x))^2] < f(x)f'(x) + \frac{1}{2}f'(x)f'''(x) \leq \frac{3}{2}f(x)f'(x).$$

Applying the fact again, we get

$$\frac{1}{2}f'(x)f''(x) < \frac{3}{4}f(x)^2.$$

Multiplying both sides by $f'(x)$ and applying the fact once more, we get

$$\frac{1}{6}(f'(x))^3 < \frac{1}{4}f(x)^3.$$

From this we deduce $f'(x) < (3/2)^{1/3}f(x) < 2f(x)$, as desired.

Note: I don’t know what the best constant is, except that it is not less than 1 (because $f(x) = e^x$ satisfies the given conditions).

B–5 We claim that the eigenvalues of $A$ are 0 with multiplicity $n - 2$, and $n/2$ and $-n/2$, each with multiplicity 1. To prove this claim, define vectors $v^{(m)}$, $0 \leq m \leq n - 1$, componentwise by $(v^{(m)})_k = e^{ikm\theta}$, and note that the $v^{(m)}$ form a basis for $\mathbb{C}^n$. (If we arrange the $v^{(m)}$ into an $n \times n$ matrix, then the determinant of this matrix is a Vandermonde product which is nonzero.) Now note that

$$(Av^{(m)})_j = \sum_{k=1}^{n} \cos(j\theta + k\theta)e^{ikm\theta} = \frac{1}{2} \left( e^{ij\theta} \sum_{k=1}^{n} e^{ik(m+1)\theta} + e^{-ij\theta} \sum_{k=1}^{n} e^{ik(m-1)\theta} \right).$$

Since $\sum_{k=1}^{n} e^{ik\theta} = 0$ for integer $\ell$ unless $n | \ell$, we conclude that $Av^{(m)} = 0$ for $m = 0$ or for $2 \leq m \leq n - 1$. In addition, we find that $(Av^{(1)})_j = \frac{n}{2}e^{-ij\theta} = \frac{n}{2}(v^{(n-1)})_j$ and $(Av^{(n-1)})_j = \frac{n}{2}e^{ij\theta} = \frac{n}{2}(v^{(1)})_j$, so that $A(v^{(1)} \pm v^{(n-1)}) = \pm \frac{n}{2}(v^{(1)} \pm v^{(n-1)})$. Thus $\{v^{(0)}, v^{(1)}, v^{(2)}, \ldots, v^{(n-1)}, v^{(1)} + v^{(n-1)}, v^{(1)} - v^{(n-1)}\}$ is a basis for $\mathbb{C}^n$ of eigenvectors of $A$ with the claimed eigenvalues.

Finally, the determinant of $I + A$ is the product of $(1 + \lambda)$ over all eigenvalues $\lambda$ of $A$; in this case, $\det(I + A) = (1 + n/2)(1 - n/2) = 1 - n^2/4$.

B–6 Choose a sequence $p_1, p_2, \ldots$ of primes as follows. Let $p_1$ be any prime dividing an element of $S$. To define $p_{j+1}$ given $p_1, \ldots, p_j$, choose an integer $N_j \in S$ relatively prime to $p_1 \cdots p_j$ and let $p_{j+1}$ be a prime divisor of $N_j$, or stop if no such $N_j$ exists.

Since $S$ is finite, the above algorithm eventually terminates in a finite sequence $p_1, \ldots, p_k$. Let $m$ be the smallest integer such that $p_1 \cdots p_m$ has a divisor in $S$. (By the assumption on $S$ with $n = p_1 \cdots p_k$, $m = k$ has this property, so $m$ is well-defined.) If $m = 1$, then $p_1 \in S$, and we are done, so assume $m \geq 2$. Any divisor $d$ of $p_1 \cdots p_m$ in $S$ must be a multiple of $p_m$, or else it would also be a divisor of $p_1 \cdots p_{m-1}$, contradicting the choice of $m$. But now $\gcd(d, N_{m-1}) = p_m$, as desired.