Solutions to the Fifty-Ninth William Lowell Putnam Mathematical Competition

Saturday, December 5, 1998

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A–1 Consider the plane containing both the axis of the cone and two opposite vertices of the cube's bottom face. The cross section of the cone and the cube in this plane consists of a rectangle of sides s and $s\sqrt{2}$ inscribed in an isosceles triangle of base 2 and height 3, where s is the side-length of the cube. (The $s\sqrt{2}$ side of the rectangle lies on the base of the triangle.) Similar triangles yield $s/3 = (1 - s\sqrt{2}/2)/1$, or $s = (9\sqrt{2} - 6)/7$.

A–2 First solution: to fix notation, let A be the area of region $DEFG$, and B be the area of $DEIH$; further let C denote the area of sector ODE, which only depends on the arc length of s. If $[XYZ]$ denotes the area of triangle $[XYZ]$, then we have $A = C + [OEG] - [ODF]$ and $B = C + [ODH] - [OEI]$. But clearly $[OEG] = [OEI]$ and $[ODF] = [ODH]$, and so $A + B = 2C$.

Second solution: We may parametrize a point in s by any of x, y, or $\theta = \tan^{-1}(y/x)$. Then A and B are just the integrals of $y\,dx$ and $x\,dy$ over the appropriate intervals; thus $A + B$ is the integral of $x\,dy - y\,dx$ (minus because the limits of integration are reversed). But $d\theta = x\,dy - y\,dx$, and so $A + B = \Delta\theta$ is precisely the radian measure of s. (Of course, one can perfectly well do this problem by computing the two integrals separately. But what's the fun in that?)

A-3 If at least one of $f(a)$, $f'(a)$, $f''(a)$, or $f'''(a)$ vanishes at some point a, then we are done. Hence we may assume each of $f(x)$, $f'(x)$, $f''(x)$, and $f'''(x)$ is either strictly positive or strictly negative on the real line. By replacing $f(x)$ by $-f(x)$ if necessary, we may assume $f''(x) > 0$; by replacing $f(x)$ by $f(-x)$ if necessary, we may assume $f'''(x) > 0$. (Notice that these substitutions do not change the sign of $f(x)f'(x)f''(x)f'''(x)$.) Now $f''(x) > 0$ implies that $f'(x)$ is increasing, and $f'''(x) > 0$ implies that $f'(x)$ is convex, so that $f'(x + a) > f'(x) + af''(x)$ for all x and a. By letting a increase in the latter inequality, we see that $f'(x + a)$ must be positive for sufficiently large a; it follows that $f'(x) > 0$ for all x. Similarly, $f'(x) > 0$ and $f''(x) > 0$ imply that $f(x) > 0$ for all x. Therefore $f(x)f'(x)f''(x)f'''(x) > 0$ for all x, and we are done.

A-4 The number of digits in the decimal expansion of A_n is the Fibonacci number F_n, where $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. It follows that the sequence $\{A_n\}$, modulo 11, satisfies the recursion $A_n = (-1)^{F_{n-2}} A_{n-1} + A_{n-2}$. (Notice that the recursion for A_n depends only on the value of F_{n-2} modulo 2.) Using these recursions, we find that $A_7 \equiv 0$ and $A_8 \equiv 1$ modulo 11, and that $F_7 \equiv 1$ and $F_8 \equiv 1$ modulo 2. It follows that $A_n \equiv A_{n+6} \pmod{11}$ for all $n \geq 1$. We find that among A_1, A_2, A_3, A_4, A_5, and A_6, only A_1 vanishes modulo 11. Thus 11 divides A_n if and only if $n = 6k + 1$ for some nonnegative integer k.

A–5 Define the sequence D_i by the following greedy algorithm: let D_1 be the disc of largest radius (breaking ties arbitrarily), let D_2 be the disc of largest radius not meeting D_1, let D_3 be the disc of largest radius not meeting D_1 or D_2, and so on, up to some final disc D_n. To see that $E \subseteq \cup_{j=1}^n 3D_j$, consider a point in E; if it lies in one of the D_i, we are done. Otherwise, it lies in a disc D of radius r, which meets one of the D_i having radius $s \geq r$ (this is the only reason a disc can be skipped in our algorithm). Thus the centers lie at a distance $t < s + r$, and so every point at distance less than r from the center of D lies at distance at most $r + t < 3s$ from the center of the corresponding D_i.

A–6 Recall the inequalities $|AB|^2 + |BC|^2 \geq 2|AB||BC|$ (AM-GM) and $|AB||BC| \geq 2[ABC]$ (Law of Sines). Also recall that the area of a triangle with integer coordinates is half an integer (if its vertices lie at $(0,0), (p,q), (r,s)$, the area is $|ps - qr|/2$), and that if A and B have integer coordinates, then $|AB|^2$ is an integer (Pythagoras). Now observe that

$$8[ABC] \leq |AB|^2 + |BC|^2 + 4[ABC] \leq |AB|^2 + |BC|^2 + 2|AB||BC| < 8[ABC] + 1,$$

and that the first and second expressions are both integers. We conclude that $8[ABC] = |AB|^2 + |BC|^2 + 4[ABC]$, and so $|AB|^2 + |BC|^2 = 2|AB||BC| = 4[ABC]$; that is, B is a right angle and $AB = BC$, as desired.

B–1 Notice that

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)} = (x + 1/x)^3 - (x^3 + 1/x^3) = 3(x + 1/x)$$

(difference of squares). The latter is easily seen (e.g., by AM-GM) to have minimum value 6 (achieved at $x = 1$).

B–2 Consider a triangle as described by the problem; label its vertices A, B, C so that $A = (a, b)$, B lies on the x-axis, and C lies on the line $y = x$. Further let $D = (a, -b)$ be the reflection of A in the x-axis, and let $E = (b, a)$ be the reflection of A in the line $y = x$. Then $AB = DB$ and $AC = CE$, and so the perimeter of ABC is $DB + BC + CE \geq DE = \sqrt{(a-b)^2 + (a+b)^2} = \sqrt{2a^2 + 2b^2}$. It is clear that this lower bound can be achieved; just set B (resp. C) to be the intersection between the segment DE and the x-axis (resp. line $x = y$); thus the minimum perimeter is in fact $\sqrt{2a^2 + 2b^2}$.

B–3 We use the well-known result that the surface area of the "sphere cap" $\{(x,y,z)\,|\,x^2+y^2+z^2=1,\ z\geq z_0\}$ is simply $2\pi(1-z_0)$. (This result is easily verified using calculus; we omit the derivation here.) Now the desired surface area is just 2π minus the surface areas of five identical halves of sphere caps; these caps, up to isometry, correspond to z_0 being the distance from the center of the pentagon to any of its sides, i.e., $z_0=\cos\frac{\pi}{5}$. Thus the desired area is $2\pi-\frac{5}{2}\left(2\pi(1-\cos\frac{\pi}{5})\right)=5\pi\cos\frac{\pi}{5}-3\pi$ (i.e., $B=\pi/2$).

B–4 For convenience, define $f_{m,n}(i)=\lfloor\frac{i}{m}\rfloor+\lfloor\frac{i}{n}\rfloor$, so that the given sum is $S(m,n)=\sum_{i=0}^{mn-1}(-1)^{f_{m,n}(i)}$. If m and n are both odd, then $S(m,n)$ is the sum of an odd number of ±1's, and thus cannot be zero. Now consider the case where m and n have opposite parity. Note that $\lfloor\frac{i}{m}\rfloor+\lfloor k-\frac{i+1}{m}\rfloor=k-1$ for all integers i,k,m. Thus $\lfloor\frac{i}{m}\rfloor+\lfloor\frac{mn-i-1}{m}\rfloor=n-1$ and $\lfloor\frac{i}{n}\rfloor+\lfloor\frac{mn-i-1}{n}\rfloor=m-1$; this implies that $f_{m,n}(i)+f_{m,n}(mn-i-1)=m+n-2$ is odd, and so $(-1)^{f_{m,n}(i)}=-(-1)^{f_{m,n}(mn-i-1)}$ for all i. It follows that $S(m,n)=0$ if m and n have opposite parity.

Now suppose that $m=2k$ and $n=2l$ are both even. Then $\lfloor\frac{2j}{2m}\rfloor=\lfloor\frac{2j+1}{2m}\rfloor$ for all j, so S can be computed as twice the sum over only even indices:

$$S(2k,2l)=2\sum_{i=0}^{2kl-1}(-1)^{f_{k,l}(i)}=S(k,l)(1+(-1)^{k+l}).$$

Thus $S(2k,2l)$ vanishes if and only if $S(k,l)$ vanishes (if $1+(-1)^{k+l}=0$, then k and l have opposite parity and so $S(k,l)$ also vanishes).

Piecing our various cases together, we easily deduce that $S(m,n)=0$ if and only if the highest powers of 2 dividing m and n are different.

B–5 Write $N=(10^{1998}-1)/9$. Then

$$\sqrt{N}=\frac{10^{999}}{3}\sqrt{1-10^{-1998}}=\frac{10^{999}}{3}(1-\frac{1}{2}10^{-1998}+r),$$

where $r<10^{-2000}$. Now the digits after the decimal point of $10^{999}/3$ are given by $.3333\ldots$, while the digits after the decimal point of $\frac{1}{6}10^{-999}$ are given by $.00000\ldots1666666\ldots$. It follows that the first 1000 digits of $\sqrt{N}$ are given by $.33333\ldots3331$; in particular, the thousandth digit is 1.

B–6 First solution: Write $p(n)=n^3+an^2+bn+c$. Note that $p(n)$ and $p(n+2)$ have the same parity, and recall that any perfect square is congruent to 0 or 1 (mod 4). Thus if $p(n)$ and $p(n+2)$ are perfect squares, they are congruent mod 4. But $p(n+2)-p(n)\equiv 2n^2+2b \pmod 4$, which is not divisible by 4 if n and b have opposite parity.

Second solution: We prove more generally that for any polynomial $P(z)$ with integer coefficients which is not a perfect square, there exists a positive integer n such that $P(n)$ is not a perfect square. Of course it suffices to assume $P(z)$ has no repeated factors, which is to say $P(z)$ and its derivative $P'(z)$ are relatively prime.

In particular, if we carry out the Euclidean algorithm on $P(z)$ and $P'(z)$ without dividing, we get an integer D (the discriminant of P) such that the greatest common divisor of $P(n)$ and $P'(n)$ divides D for any n. Now there exist infinitely many primes p such that p divides $P(n)$ for some n: if there were only finitely many, say, $p_1, \dots, p_k$, then for any n divisible by $m = P(0)p_1p_2\cdots p_k$, we have $P(n) \equiv P(0) \pmod{m}$, that is, $P(n)/P(0)$ is not divisible by $p_1, \dots, p_k$, so must be ± 1, but then P takes some value infinitely many times, contradiction. In particular, we can choose some such p not dividing D, and choose n such that p divides $P(n)$. Then $P(n+kp) \equiv P(n) + kpP'(n) (\bmod\, p)$ (write out the Taylor series of the left side); in particular, since p does not divide $P'(n)$, we can find some k such that $P(n+kp)$ is divisible by p but not by p^2, and so is not a perfect square.

Third solution: (from David Rusin, David Savitt, and Richard Stanley independently) Assume that $n^3 + an^2 + bn + c$ is a square for all $n > 0$. For sufficiently large n,

$$(n^{3/2} + \frac{1}{2}an^{1/2} - 1)^2 < n^3 + an^2 + bn + c < (n^{3/2} + \frac{1}{2}an^{1/2} + 1)^2;$$

thus if n is a large even perfect square, we have $n^3 + an^2 + bn + c = (n^{3/2} + \frac{1}{2}an^{1/2})^2$. We conclude this is an equality of polynomials, but the right-hand side is not a perfect square for n an even non-square, contradiction. (The reader might try generalizing this approach to arbitrary polynomials. A related argument, due to Greg Kuperberg: write $\sqrt{n^3 + an^2 + bn + c}$ as $n^{3/2}$ times a power series in $1/n$ and take two finite differences to get an expression which tends to 0 as $n \to \infty$, contradiction.)

Note: in case $n^3 + an^2 + bn + c$ has no repeated factors, it is a square for only finitely many n, by a theorem of Siegel; work of Baker gives an explicit (but large) bound on such n. (I don't know whether the graders will accept this as a solution, though.)