First $\star$ problem

If a and b are elements of a ring, define $[a,b] = ab - ba$ and inductively $a^{(k)} = [a^{(k-1)}, b]$ (note that for the sake of simplicity we do not indicate the dependence of $a^{(k)}$ on b). Prove the following formula

$$\sum_{i=0}^{k} b^i a b^{k-i} = \sum_{j=0}^{k} \binom{k+1}{j+1} b^{k-j} a^{(j)}$$

Proof: The $k=0$ case is clearly true since $a = a^{(0)}$.

Suppose the formula is true for k: show that it is true for $k+1$.

$$\begin{aligned}
\sum_{i=0}^{k+1} b^i a b^{k+1-i} &= (\sum_{i=0}^{k} b^i a b^{k-i})b + b^{k+1}a \\
&= (\sum_{j=0}^{k} \binom{k+1}{j+1} b^{k-j} a^{(j)})b + b^{k+1}a \\
&= \sum_{j=0}^{k} \binom{k+1}{j+1} b^{k-j}[a^{(j+1)} + ba^{(j)}] + b^{k+1}a \\
&= \sum_{j'=1}^{k+1} \binom{k+1}{j'} b^{k+1-j'} a^{(j')} + \sum_{j=0}^{k} \binom{k+1}{j+1} b^{k+1-j} a^{(j)} \\
&= (k+1)b^{k+1}a + \sum_{j=1}^{k} [\binom{k+1}{j} + \binom{k+1}{j+1}] b^{k+1-j} a^{(j)} + a^{(k+1)} + b^{k+1}a \\
&= (k+2)b^{k+1}a + \sum_{j=1}^{k} \binom{k+2}{j+1} b^{k+1-j} a^{(j)})b + a^{(k+1)} \\
&= \sum_{j=0}^{k+1} \binom{k+2}{j+1} b^{k+1-j} a^{(j)}
\end{aligned}$$