TEST 2

Davis
M235

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

(12pts.) **1.** Find the second order Taylor formula for $f(x,y) = e^{x+2y}$ about the point $(x_0, y_0) = (0,0)$. Use the formula to estimate $f(.02, -.03)$, and compare the estimate to the actual value.

We need to take all the partial derivatives and plug in the point (0,0) into those partials. We get:

$$f_x = e^{x+2y}; f_y = 2e^{x+2y}; f_{xx} = e^{x+2y}; f_{xy} = 2e^{x+2y}; f_{yy} = 4e^{x+2y}.$$

This implies that the Taylor polynomial is $T(h_1, h_2) = 1 + h_1 + 2h_2 + \frac{1}{2}(h_1^2 + 4h_1h_2 + 4h_2^2)$. To get the approximation, we see that $f(.02, -.03) = e^{-.04} = .9607894 \cong T(.02, -.03) = 1 + .02 - 2(-.03) + \frac{1}{2}((.02)^2 + 4(.02)(-.03) + 4(-.03)^2) = .9608$.

(14pts.) **2.** Find the critical points of $f(x,y) = x^3/3 - 2xy + y^2 - 3x$, and classify them as local maximum, local minimum, or saddle points.

To find critical points, we set the first partials equal to 0 and solve for x and y. Thus, $f_x = x^2 - 2y - 3 = 0$ and $f_y = -2x + 2y = 0$. The second equation implies that $y = x$, so substituting that into the first equation yields $x^2 - 2x - 3 = 0$. This has solutions $x = -1, 3$, so the critical points are $(x,y) = (-1,-1)$ or $(3,3)$. We need to get the second partials to determine whether these are max, min, or saddle points. $f_{xx} = 2x; f_{xy} = -2; f_{yy} = 2$, implying that the Hessian for the critical point $(-1,-1)$ is -8, which implies a saddle point. The Hessian for the critical point $(3,3)$ is 12, and since $f_{xx} > 0$ for this critical point $(3,3)$ is a local min.

(12pts.) **3.** Show that the rectangular box of given volume has minimum surface area when the box is a cube.

Call the sides of the box x, y, and z, and call the fixed volume $V = xyz$. The surface area S is $S = 2xy + 2xz + 2yz$. We plug $z = \frac{V}{xy}$ into this equation to yield $S = 2xy + \frac{2V}{y} + \frac{2V}{x}$. Take the partials of S and set them equal to 0: $f_x = 2y - \frac{2V}{x^2} = 0$ and $f_y = 2x - \frac{2V}{y^2} = 0$. Solving the first equation for y and plugging it into the second yields $2x - \frac{2x^4}{V} = 0$, implying that either $x = 0$ or $x = V^{\frac{1}{3}}$. This leads to $y = V^{\frac{1}{3}}$ and $z = V^{\frac{1}{3}}$ as the only critical point ($x = 0$ doesn't give us a box), and the Hessian is 12 implying that this critical point is a max. Since all three dimensions are the same, the box is a cube as claimed.

(12pts.) **4.** Argue that for a function $F(x,y,z) = x^2yzi + e^{xyz}j + \sin(x+y+z)k$, $divcurlF = 0$ (show your computations).

We first compute the curl: $curlF = i(\cos(x+y+z) - xye^{xyz}) - j(\cos(x+y+z) - x^2y) + k(yze^{xyz} - x^2z)$. We then take the div of this: $divcurlF = (-\sin(x+y+z) - xy^2ze^{xyz} - ye^{xyz}) - (-\sin(x+y+z) - x^2) + (xy^2ze^{xyz} - x^2) = 0$.

(30pts.) **5.** Do the the following problems.

a. $\int_0^1 \int_y^1 e^{x^2} dxdy$ (HINT: Switching the order of integration may help!)

Switching the order of integration yields $\int_0^1 \int_0^x e^{x^2} dydx = \int_0^1 xe^{x^2} dx = \frac{1}{2}e^{x^2}|_0^1 = \frac{1}{2}(e-1)$.

b. Find the volume of the region common to the intersecting cylinders $x^2+y^2 \leq 4$ and $x^2+z^2 \leq 4$.

Integrating with respect to z first has the $x^2+z^2=4$ cylinder as the top and bottom, so the limits will be $z = \pm\sqrt{4-x^2}$. The shadow that is cast into the xy-plane is the circle determined by the other cylinder, namely $x^2+y^2=4$, so we can solve this out for y first and then x. To find a volume we do a triple integral of the function 1:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 dzdydx = 2\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2} dydx = 4\int_{-2}^{2} (4-x^2)dx = 4(4x-x^3/3)$$

c. $\int_0^2 \int_0^{6-3x} \int_0^{6-3x-2y} \sqrt{z} dzdydx$ (Hint: switching the order of integration might make your life easier here).

You can do this problem as it is, but it is a little easier if you do the x integration first, then y, with the z integration last. The region for this integral is essentially under the plane $3x+2y+z=6$, so solve that for $x = 2-\frac{2}{3}y-\frac{1}{3}z$. The shadow in the yz plane is $2y+z=6$, so integrate from $y=0$ to $y=\frac{6-z}{2}$ and then from $z=0$ to $z=6$. This yields the following computations:

$$\int_0^6 \int_0^{\frac{6-z}{2}} \int_0^{2-\frac{2}{3}y-\frac{1}{3}z} \sqrt{z} dxdydz = \cdots = \int_0^6 \frac{(6-z)^2}{12}\sqrt{z} dz = \cdots = \frac{96\sqrt{6}}{35}$$

(20pts.) **6.** The average value of a function $f(x,y,z)$ on a region W is defined to be the triple integral $f_{ave} = \frac{\int\int\int_W f(x,y,z)dzdydx}{\int\int\int_W dzdydx}$. Find the average value of the function $f(x,y,z) = x^2+y^2+z^2$ over the region bounded by $x+y+z=4, x=0, y=0$, and $z=0$.

If we integrate with respect to z first, the limits of integration will be $z=0$ to $z = 4-x-y$. If we then do y, the limits will be from $y=0$ to $y=4-x$, and finally x will go from 0 to 4. This will be true for both integrals. In the numerator integral we get:

$$\int_0^4 \int_0^{4-x} \int_0^{4-x-y} (x^2+y^2+z^2)dzdydx = \cdots = \int_0^4 (\frac{x^2(4-x)^2}{2} + \frac{(4-x)^4}{6})dx = \cdots = \frac{256}{5}$$