TEST 1

Davis
M212

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

(30pts.) **1.** Consider the triangle with vertices $A = (1,2,3), B = (-1,4,-5)$, and $C = (2,2,2)$.

$AB = (2,-2,8); AC = (-1,0,1); BC = (3,-2,7)$

a. Find the lengths of the sides of the triangle.

$\|AB\| = \sqrt{72}; \|AC\| = \sqrt{2}; \|BC\| = \sqrt{62}$

b. Find the projection of AB on AC.

$\frac{AB \cdot AC}{AC \cdot AC} AC = \frac{6}{2}(-1,0,1) = (-3,0,3)$

c. Find the angle between BA and BC.

$BA \cdot BC = \|BA\|\|BC\| \cos(\theta); 66 = \sqrt{72}\sqrt{62}\cos(\theta); \theta = \cos^{-1}(\frac{66}{\sqrt{72 \cdot 62}}) \cong 8.9$ degrees.

d. Compute the area of ABC.

Area $= \frac{1}{2}\|AB \times BC\| = \frac{1}{2}\|AB\|\|BC\| \sin(\theta) = \frac{1}{2}\sqrt{72}\sqrt{62}\sin(8.9) \cong 5.2$.

e. Find the equation of the plane containing the triangle ABC.

The normal to this plane is $AB \times AC$, which is $\begin{vmatrix} i & j & k \\ 2 & -2 & 8 \\ -1 & 0 & 1 \end{vmatrix} = -2i - 10j - 2k$.

One point on the plane is $(1,2,3)$, so the plane is $-2(x-1)-10(y-2)-2(z-3) = 0$.

f. Find the parametric equation of the line determined by intersecting the plane found in part e. with the plane $-x + 2y - 3z = 4$.

This is the trickiest problem on the test. The idea is that the cross product of the normal vectors to each plane will be normal to the normals, and hence will be on both planes. Thus, this vector points in the direction of the line of intersection.

The line will be in the direction of $(-2,-10,-2) \times (-1,2,-3) = \begin{vmatrix} i & j & k \\ -2 & -10 & -2 \\ -1 & 2 & -3 \end{vmatrix} =$ $34i - 4j - 14k$. The point $(0,2,0)$ is on both planes, so $x = 34t; y = 2-4t; z = -14t$ is the parametric equation of the line.

(10pts.) **2.** Change the point $(1,-1,1)$ from rectangular coordinates to cylindrical and spherical coordinates.

For cylindrical, $r = \sqrt{2}, \theta = \tan^{-1}(-1) = \frac{-\pi}{4}$, so the cylindrical coordinates are $(\sqrt{2}, \frac{-\pi}{4}, 1)$.

For spherical, $\rho = \sqrt{3}, \theta = \frac{-\pi}{4}$, and $\phi = \cos^{-1}(\frac{1}{\sqrt{3}})$.

(15pts.) **3.** Sketch a graph of $x + y^2 - z^2 = 0$, explaining the different steps you take in making this drawing.

I want to see level curves for some choice of variables. If you choose level curves for z, then the level curves will be parabolas. If you choose level curves for x, then the level curves will be hyperbolas. In any case, you should compute *at least* 4 level curves to get the idea of what is going on, then try to graph those on the xyz axes. The pictures are very difficult to draw, but I wanted to see your approach.

(15pts.) **4.** At what point does the tangent plane to the surface $z = x^2 + 2y^3$ at $(1, 1, 3)$ meet the z-axis?

The tangent plane is $z - 3 = 2(x - 1) + 6(y - 1)$. When $x = y = 0$, we get $z = -5$, and that is the point where the tangent plane meets the z-axis.

(15pts.) **5.** Suppose that a duck is swimming in the circle $x = \cos(t), y = \sin(t)$ and that the water temperature is given by the formula $T = xy^3 + yx^3$. Find $\frac{dT}{dt}$, the rate of change in the temperature the duck might feel: (a) by the chain rule; (b) by expressing T in terms of t and differentiating.

We first find $\frac{dT}{dt}$ by writing the formula $T = \cos(t)\sin(t)(\sin^2(t)+\cos^2(t)) = \cos(t)\sin(t)$ and differentiating, yielding $\frac{dT}{dt} = -\sin^2(t) + \cos^2(t)$.

The second approach is to calculate the gradient of f, $\nabla f = (y^3+3x^2y, 3xy^2+x^3)$, evaluate that at $(\cos(t), \sin(t))$, yielding $(\sin^3(t)+3\cos^2(t)\sin(t), \cos^3(t)+3\sin^2(t)\cos(t))$, and dot that with the derivative of the path, namely $(-\sin(t), \cos(t))$ using the chain rule. This gives the same answer as the first part.

(15pts.) **6.** Explain why the direction of the gradient ∇f is the direction along which f is increasing the fastest.

If we take unit vectors u to determine the directional derivative, then the largest possible value of $\nabla f \cdot u = \|\nabla f\|\|u\|\cos(\theta) = \|\nabla f\|\cos(\theta)$ is when $\theta = 0$ (since cos is maximized at 1 when the angle is 0). This implies that the direction of maximal increase is in the same direction as the gradient as claimed.