TEST 3

Davis
CS222

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

(15pts.) **1.** In how many ways can we select a committee of four Republicans, three Democrats, and two Independents from a group of 10 distinct Republicans, 12 distinct Democrats, and 4 distinct Independents? Once you have chosen the committee, how many ways are there to choose a president, vice president, secretary, and treasurer from committee members?

$C(10,4) \times C(12,3) \times C(4,2) = 210 \times 220 \times 6 = 277,200$ ways to choose the committee. Once the committee is chosen, there are $P(9,4) = 9 \times 8 \times 7 \times 6 = 3024$ ways to choose the officers.

(10pts.) **2.** In how many ways can three nonnegative integers be added to get 7? Explain your reasoning.

You can view this as a string of 9 blank spaces, and we will put $|$ in two of the spaces to indicate where to break the numbers up. For example, $1|1111|11$ would indicate $1 + 4 + 2 = 7$. There are $C(9,2) = 36$ ways to choose where to put the $|$. (NOTE: I was accepting other answers since I did not indicate whether order mattered. In my explanation, the order does matter.)

(15pts.) **3.** Show that $C(n,k) + C(n,k+1) = C(n+1,k+1)$ for $0 \leq k \leq n-1$.

$C(n,k) + C(n,k+1) = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-(k+1))!} = \frac{n!}{k!(n-(k+1)!}(\frac{1}{n-k} + \frac{1}{k+1}) = \frac{n!}{k!(n-(k+1)!}(\frac{k+1+n-k}{(n-k)(k+1)}) = \frac{(n+1)!}{(n-k)!(k+1)!} = C(n+1,k+1)$

(10pts.) **4.** Use the binomial theorem to show that $\sum_{k=0}^{n} 2^k C(n,k) = 3^n$ (show your work).

The binomial theorem states that $(a+b)^n = \sum_{k=0}^{n} C(n,k)a^k b^{n-k}$. If we plug in $a = 2$ and $b = 1$, we get $(2+1)^n = \sum_{k=0}^{n} C(n,k)2^k 1^{n-k}$, which is the claimed result.

(10pts.) **5.** Show that any simple connected graph (no loops, no multiple edges between the same pair of vertices) must have two vertices of the same degree.

Each vertex must have degree at least 1 (since the graph is connected) and at most $(n-1)$ (since the graph is simple). There are n vertices, so the pigeonhole principle implies that there are two vertices that have the same degree.

(10pts.) **6.** Solve the recurrence relation $a_n = 2a_{n-1} + 8a_{n-2}$ with initial conditions $a_0 = 4, a_1 = 10$. Verify that your formula works for $n = 3$.

Assuming that $a_n = \alpha^n$, we get $\alpha^n = 2\alpha^{n-1} + 8\alpha^{n-2}$. This implies that $\alpha^2 - 2\alpha - 8 = 0$, or $(\alpha-4)(\alpha+2) = 0$, so $\alpha = 4$ or $\alpha = -2$. We write $a_n = C_1 4^n + C_2(-2)^n$. The initial conditions imply that $a_0 = 4 = C_1 + C_1$ and $a_1 = 10 = 4C_1 - 2C_1$. Multiply the first equation by 2 and add, yielding $18 = 6C_1$, so $C_1 = 3$, which implies that $C_2 = 1$. Thus, $a_n = 3 \cdot 4^n + (-2)^n$. To verify the $n = 3$ case, the formula we just developed implies $a_3 = 3 \cdot 4^3 + (-2)^3 = 192 - 8 = 184$. The recursive way to find a_3 is to get $a_2 = 2a_1 + 8a_0 = 2(10) + 8(4) = 52; a_3 = 2a_2 + 8a_3 = 2(52) + 8(10) = 184$.

(15pts.) **7.** How many edges are contained in $K_{r,s}$, the complete bipartite graph on $r + s$ edges? What conditions on r and s will allow an Eulerian cycle in this graph? What conditions will allow a Hamiltonian cycle?

The number of edges is rs; $K_{r,s}$ will allow an Eulerian cycle whenever the degrees of all of the vertices are even, which happens when r and s are both even. For a Hamiltonian cycle, you need $r = s$ (otherwise, you would not get back to where you started).

(15pts.) **8.** Dijkstra's algorithm is the following:

```
procedure dijkstra(w,a,z,L)
L(a) := 0
for all vertices x ≠ a do
L(x) := ∞
T:=set of all vertices
while z ∈ T do
begin
choose v ∈ T with minimum L(v)
T := T − {v}
for each x ∈ T adjacent to v do
```

11. $L(x) := \min\{L(x), L(v) + w(v, x)\}$

12. end

13. end dijkstra

Show how this algorithm can be used to trace the shortest path through the graph listed below.

See the book for examples of this.