# TEST 3

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M212

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

(20pts.) **1. a.** Find the the MacLaurin series for $e^{-x^2}$. What is its interval of convergence?

**b.** Use the answer from part a. to approximate $\int_0^{.2} e^{-x^2}dx$ within .001. Justify your answer!

**a.** The MacLaurin series for $e^x$ is $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$, and it has an interval of convergence of all real numbers. To get the MacLaurin series for $e^{-x^2}$, we plug $-x^2$ in wherever we see an $x$ in the power series for $e^x$, yielding $1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots+\frac{(-x^2)^n}{n!}+\cdots$.

**b.** To estimate $\int_0^{.2} e^{-x^2}dx$ within .001, we substitute the power series in for $e^{-x^2}$ to get $\int_0^{.2}(1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots+\frac{(-x^2)^n}{n!}+\cdots)dx = x-\frac{x^3}{3}+\frac{x^5}{10}-\frac{x^7}{42}+\cdots|_0^{.2} = .2-\frac{(.2)^3}{3}+\frac{(.2)^5}{10}+\cdots$. This series is alternating and converges, so we can estimate its sum by taking the partial sum with the property that the next term we would have added is smaller than our target accuracy. Since $\frac{(.2)^5}{42} < .001$, we can add the first two terms to get our estimate, and that is $.2-\frac{(.2)^3}{3} = .197333$.

(20pts.) **2. a.** Compute the first four nonzero terms of the MacLaurin series for the function $f(x) = (1+x)^{-\frac{1}{3}}$.

**b.** Estimate $(1.1)^{-\frac{1}{3}}$ within .1, and justify your answer.

**a.** The MacLaurin recipe tells us to take derivatives and plug in 0, so we get $f(x) = (1+x)^{-\frac{1}{3}}$; $f(0) = 1$; $f'(x) = -\frac{1}{3}(1+x)^{-\frac{4}{3}}$; $f'(0) = -\frac{1}{3}$; $f''(x) = -\frac{1}{3}(-\frac{4}{3})(1+x)^{-\frac{7}{3}}$; $f''(0) = \frac{4}{9}$; $f'''(x) = -\frac{1}{3}(-\frac{4}{3})(-\frac{7}{3})(1+x)^{-\frac{10}{3}}$; $f'''(0) = -\frac{28}{27}$. Putting these values into the sum $\Sigma_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$, we get that the first four nonzero terms of the MacLaurin series for $(1+x)^{-\frac{1}{3}}$ are $1-\frac{1}{3}x+\frac{2}{9}x^2-\frac{14}{81}x^3+\cdots$.

**b.** To estimate $(1.1)^{-\frac{1}{3}}$, we plug $x = .1$ into the MacLaurin series for $(1+x)^{-\frac{1}{3}}$. We get $1-\frac{1}{3}(.1)+\frac{2}{9}(.1)^2-\frac{14}{81}(.1)^3+\cdots$. This series is alternating, so we can estimate the sum by stopping just before the term that is smaller than the target accuracy. In this case, the second term is smaller than .1, so our estimate is just the first term which is 1 (there were other acceptable answers, but I wanted to see your reasoning). The true answer on a calculator is .9687923.

(20pts.) **3.** Rabbits ($x$) and Wolves ($y$) satisfy the following differential equations:

$$\frac{dx}{dt} = .05x - .001xy;$$
$$\frac{dy}{dt} = -.1y + .001y^2 + .0001xy$$

**a.** Find equilibrium solutions to this system.

**b.** Find a differential equation for $\frac{dy}{dx}$.

**a.** To find equilibrium solutions, we set both derivatives equal to 0. The first equation becomes $x(.05 - .001y) = 0$, so either $x = 0$ or $.05 - .001y = 0$. If $x = 0$, then we look at the second equation and it simplifies to $y(-.1+.001y+.0001(0)y) = 0$. Either $y = 0$ or $-.1+.001y = 0$, so either $y = 0$ or $y = 100$. Thus, we get two equilibrium solutions from this case: $(x, y) = (0, 0)$ or $(x, y) = (0, 100)$. If $.05 - .001y = 0$, then $y = 50$. Plugging that into the second equation yields $50(-.1 + .001(50) + .0001x(50)) = 0$. Solving for $x$ leads to the equilibrium solution $(x, y) = (500, 50)$. Thus, there are three equilibrium solutions.

**b.** $\frac{dy}{dx} = \frac{-.1y+.001y^2+.0001xy}{.05x-.001xy}$ (we used the chain rule for this, putting $\frac{dy}{dt}$ over $\frac{dx}{dt}$).

**c.** We only need to compute the slope of the tangent line at the point $x = 400, y = 40$, which we do by plugging those numbers into the answer to part b. We get $\frac{dy}{dx} = \frac{-.1(40)+.001(40)^2+.0001(400)(40)}{.05(400)-.001(400)(40)} = -.2$. We can use this in Euler's method to get an estimate of the wolf population: $y_{new} = m(\text{step size}) + y_{old} = (-.2)(25) + 40 = 35$.

(20pts.) **4.** Consider the differential equation $\frac{dy}{dx} = y(1 - x/3)$.

**a.** Sketch the direction field associated to the differential equation. You should show lines at integer points for $x$ and $y$ between 0 and 5.

**b.** Use Euler's method with a step size of $h = 1$ to estimate the value of $y(2)$, where $y(0) = 1$.

**c.** Solve the differential equation and use the initial value of $y(0) = 1$ to get an exact value for $y(2)$.

**a.** The key features I looked for (in the first quadrant) on the direction field were (i) horizontal tangent lines for all points on the vertical line $x = 3$; (ii) increasing tangent lines for points where $x < 3$; (iii) decreasing tangent lines for points where $x > 3$.

**b.** Euler's method yields the following: $y_1 = 1(1) + 1 = 2; y_2 = \frac{4}{3}(1) + 2 = \frac{10}{3}$.

**c.** For the exact solution we separate the variables and integrate. $\int \frac{dy}{y} = \int(1-x/3)dx; \ln(y) = x - \frac{x^2}{6} + C; y = C'e^{x-x^2/6}$. Since $y(0) = 1$, we get $1 = C'e^0$, so $C' = 1$. When $x = 2$, $y = e^{2-\frac{4}{6}} \cong 3.79$.

(20pts.) **5.** A roast turkey is taken from an oven when the temperature has reached 90 degrees centigrade and placed on a table in a room where the temperature is 20 degrees centigrade.

**a.** If the rate of cooling is proportional to the temperature difference between the object and its surroundings, write a differential equation for this Thanksgiving situation.

**b.** Solve the differential equation for temperature.

**c.** If the temperature is 70 degrees after 15 minutes, how long does it take to cool to 50 degrees?

**a.** $\frac{dT}{dt} = k(T - 20)$.

**b.** Separate the variables and solve for $T$: $\int \frac{dT}{T-20} = \int kdt; \ln(T - 20) = kt + C; T - 20 = C'e^{kt}; T = 20 + C'e^{kt}$. Since the initial temperature is 90, we get $90 = 20 + C'e^0$, or $C' = 70$.

**c.** $70 = 20 + 70e^{k(15)}; \ln(\frac{5}{7}) = 15k; k = -.0224315$. If we want to know when the temperature cools down to 50 degrees, we solve $50 = 20+70e^{-.0224315t}; \ln(\frac{3}{7}) = -.0224315t; t = 37.77$ minutes.