TEST 1

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M212

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

(21pts.) **1.** Without your calculators or the Tables, compute the following:

a. $\int_1^e \frac{(\ln(x))^3}{x} dx$

Set $u = \ln(x), du = \frac{1}{x}dx$, changing the integral to $\int_0^1 u^3 du$ (I change the limits of integration as well). This integrates to $\frac{u^4}{4}|_0^1 = \frac{1}{4}$.

b. $\int x \tan^{-1}(x^2) dx$

Set $w = x^2, dw = 2xdx$, changing the integral to $\int \tan^{-1}(w)dw$ (I used the variable w instead of u since I will be doing integration by parts). We do integration by parts on this with $u = \tan^{-1}(w), v' = 1, du = \frac{1}{1+w^2}, v = w$ to get $\int \tan^{-1}(w)dw = w\tan(w) - \int \frac{w}{1+w^2}dw$. This last integral is a basic u-substitution with $u = 1 + w^2, du = 2wdw$, yielding $\int x\tan^{-1}(x^2)dx = w\tan(w) - \int \frac{w}{1+w^2}dw = w\tan(w) - \frac{1}{2}\ln(1+w^2) + C = x^2\tan(x^2) - \frac{1}{2}\ln(1+x^4) + C$.

c. $\int \frac{x+3}{x^2+4x+3} dx$

The integrand simplifies to $\frac{x+3}{(x+3)(x+1)} = \frac{1}{x+1}$. When this is integrated we get $\ln(x+1) + C$. I intended this to be a simple partial fractions, which was fine if you did it that way, but the simplification makes it even easier.

(21pts.) **2.** Verify the formula $\int x^n e^{ax} dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx$ (a) by differentiation; and (b) by using integration by parts. Use the formula to calculate $\int x^2 e^{2x} dx$.

For part (a), we differentiate the right hand side: $(\frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx)' = \frac{1}{a}[x^n a e^{ax} + nx^{n-1}e^{ax}] - \frac{n}{a}x^{n-1}e^{ax} = x^n e^{ax} + \frac{n}{a}x^{n-1}e^{ax} - \frac{n}{a}x^{n-1}e^{ax} = x^n e^{ax}$ as required.

Part (b) is a straightforward application of integration by parts with $u = x^n, v' = e^{ax}, u' = nx^{n-1}, v = \frac{1}{a}e^{ax}$, leading to the formula $\int x^n e^{ax}dx = x^n \frac{1}{a}e^{ax} - n\int x^{n-1}\frac{1}{a}e^{ax}dx$. Moving the $\frac{1}{a}$ to the front of both terms yields the formula.

The final computation involves applying this reduction formula with $n = 2$ and $a = 2$, giving $\int x^2 e^{2x}dx = \frac{1}{2}x^2 e^{2x} - \frac{2}{2}\int x^1 e^{2x}dx$. We can apply the formula again with $n = 1, a = 2$ to get $\int x^2 e^{2x}dx = \frac{1}{2}x^2 e^{2x} - \frac{2}{2}\int x^1 e^{2x}dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}[xe^{2x} - \frac{1}{2}\int x^0 e^{2x}dx] = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$.

(21pts.) **3.** Use $f(x) = \frac{1}{x^3+1}$.

a. Use the Trapezoid Rule with $n = 3$ to estimate $\int_1^4 f(x)dx$.

The width of the interval is $\frac{4-1}{3} = 1$. Thus, the area in the trapezoids is $\frac{4-1}{3}\left(\frac{\frac{1}{1^3+1}+\frac{1}{2^3+1}}{2} + \frac{\frac{1}{2^3+1}+\frac{1}{3^3+1}}{2} + \frac{\frac{1}{3^3+1}+\frac{1}{4^3+1}}{2}\right) \cong .4$.

b. Find a bound on the error you make in the estimate in part a.

The hardest part of this is finding the value of K. To get that, we need the second derivative of $f(x) = \frac{1}{x^3+1}$, so $f'(x) = -(x^3+1)^{-2}(3x^2)$; $f''(x) = 2(x^3+1)^{-3}(3x^2) - (x^3+1)^{-2}(6x)$. We plug in the endpoints and get $f''(1) = \frac{6}{8} - \frac{6}{4} = -\frac{3}{4}$ and $f''(4)$ is a really small number, so $K = \frac{3}{4}$. Once we have this, the bound on the error is $|E_T| \le \frac{\frac{3}{4}(4-1)^3}{12 \cdot 3^2} = \frac{3}{16}$.

c. Use the comparison test to whether $\int_1^\infty f(x)dx$ converges or diverges.

I claim that $\frac{1}{x^3+1} < \frac{1}{x^3}$. You can justify this either by observing that adding 1 to the denominator makes the denominator bigger and hence the fraction smaller, or you could do the series of equations $1 > 0; x^3 + 1 > x^3; \frac{1}{x^3} > \frac{1}{x^3+1}$. Once this is established, we need to determine the convergence of the integral $\int \frac{1}{x^3}dx$. We either recognize that this is one of our $\int \frac{1}{x^p}dx$ cases with $p > 1$, which implies convergence, or you could do the integral (I leave that to you). Since the larger function has a finite area under it on the interval from 1 to ∞, the smaller function $f(x)$ must also converge.

(20pts.) **4.** Determine whether the following integrals converge or diverge.

a. $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\cos(x)}dx$

The "problem" for this integral occurs at $x = \frac{\pi}{2}$ since the denominator is equal to 0 there. Thus, we write this integral as $\lim_{t\to\frac{\pi}{2}^-}[\int_0^t \frac{\sin(x)}{\cos(x)}dx] = \lim_{t\to\frac{\pi}{2}^-}[-\ln(\cos x)|_0^t] = \lim_{t\to\frac{\pi}{2}^-}[-\ln(\cos t)+\ln 1]$. Since $\cos t \to 0^+$ as $t \to \frac{\pi}{2}^-$ and $\ln x \to -\infty$ as $x \to 0^+$, we get that this integral diverges.

b. $\int_1^\infty \frac{1}{x^{1/2}+e^{3x}}dx$

We compare this to $\int_1^\infty \frac{1}{e^{3x}}dx$. You can make the case either by saying that adding $\sqrt{x}$ to the denominator makes the denominator bigger and hence the fraction smaller or you could use the equations $\sqrt{x} > 0; e^{3x} + \sqrt{x} > e^{3x}; \frac{1}{e^{3x}} > \frac{1}{\sqrt{x}+e^{3x}}$. We then show that $\int_1^\infty e^{-3x}dx$ is finite, and we use the comparison theorem to say that when the top integral converges the bottom integral (in this case $\int_1^\infty \frac{1}{x^{1/2}+e^{3x}}dx$) must also converge.

(17pts.) **5.** Find the area between the curves $x = y^2 - 4$ and $x = -y^2 - 2y + 8$.

This is going to be a right minus left problem. We must find the points of intersection by setting the equations equal to eachother. When we do that we get $y^2 - 4 = -y^2 - 2y + 8; 2y^2 + 2y - 12 = 0; (2y+6)(y-2) = 0; y = -3$ or $y = 2$. Thus, the area between these curves is

$$\int_{-3}^2 [(-y^2-2y+8)-(y^2-4)]dy = (-\frac{2}{3}y^3-y^2+12y)_{-3}^2 = -\frac{16}{3}-4+24-(18-9-36) = 41\frac{2}{3}$$