I. Consider the following matrix:

\[ L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}. \]

**a.** Find an inverse for \( L \) in two different ways and verify that you get the same result.

The easiest way to get this inverse is to start with the matrix

\[ M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{bmatrix} \]

and row reduce it so you get the identity on the left and the inverse of \( L \) on the right: you should get \( L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix} \). You could use elementary matrices or Cramer’s rule for the other method.

**b.** If the matrix \( A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 13 \\ 6 & 9 & 32 \end{bmatrix} \) has the \( LU \)-decomposition \( A = LU \) for \( L \) from part (a) of the problem and \( U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \) (assume this is true: you do not need to verify this!!), use your answer from part (a) to determine which values of \( b \) will make the equation \( Ax = b \) inconsistent.

If you multiply both sides of \( Ax = b \) by \( L^{-1} \), you will get \( Ux = \begin{bmatrix} b_1 \\ -2b_1 + b_2 \\ 5b_1 - 4b_2 + b_3 \end{bmatrix} \).

If \( 5b_1 - 4b_2 + b_3 \neq 0 \), then the system will be inconsistent. Thus, any vector \( B \) not on that plane will produce an inconsistent system.

II. Find a basis for \( \text{Null} A \) and \( \text{Col} A \) of the following matrix. State the rank theorem and verify it for this matrix.

\[ A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \]

Row reducing this matrix yields the reduced echelon form \( E = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). The pivot columns are the first two columns, so we can use the first two columns of \( A \) as a basis for the Column Space (in other words, \( \{ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \} \) is a basis for the column space). There are lots of other answers for this. For the Null space, we use our techniques from earlier in the course to get that \( \{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \} \) is a basis for the Null Space.

The Rank Theorem states that the dimension of the column space plus the rank of the matrix is the number of columns, and in this case that is \( 2 + 2 = 4 \).
10pts. III. Let $V$ be a vector space and let $\{v_1, v_2, \ldots, v_p\}$ be a basis for $V$. Show that every element of $V$ can be written uniquely as a linear combination of the vectors $\{v_1, v_2, \ldots, v_p\}$.

Every vector can be written in at least one way as a linear combination of the vectors $\{v_1, v_2, \ldots, v_p\}$ since these vectors span the space. Suppose a vector $v$ in $V$ can be written as $v = a_1v_1 + a_2v_2 + \cdots + a_pv_p$ and $v = b_1v_1 + b_2v_2 + \cdots + b_pv_p$. Setting these two equal to each other and moving everything to one side of the equation yields $(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_p - b_p)v_p = 0$. We know that $\{v_1, v_2, \ldots, v_p\}$ are linearly independent, which by definition implies that $(a_i - b_i) = 0$ for every $i$, and hence the two ways to write $v$ were not really distinct. This shows that there is a unique way to write every vector in $V$ as a linear combination of the $\{v_1, v_2, \ldots, v_p\}$.

10pts. IV. Suppose $A$ and $B$ are $n \times n$ matrices so that the columns of $A$ span $\mathbb{R}^n$ and $B$ is row equivalent to the $n \times n$ identity matrix. Are the columns of the matrix $AB$ linearly independent? Explain your answer.

This uses the Invertible Matrix Theorem (IMT) multiple times. Since the columns of $A$ span $\mathbb{R}^n$, the IMT implies that $A$ is invertible. Since $B$ is row equivalent to the identity matrix, the IMT implies that $B$ is invertible. The product of two invertible matrices is invertible, implying that $AB$ is invertible. Finally, the IMT implies that an invertible matrix has linearly independent columns, so $AB$ does have linearly independent columns.

15pts. V. Let $V$ be the collection of polynomials of degree at most 6 that are divisible by the polynomial $x^2 + x + 1$. Either verify that $V$ is a subspace, or give a counterexample why it is not a subspace.

If we take two polynomials, say $f$ and $g$, that are divisible by $x^2 + x + 1$, then $f + g$ will also be divisible by $x^2 + x + 1$. We can see this by using the definition of divisible, which says that $f$ is divisible by $x^2 + x + 1$ if there is a $q_1(x)$ so that $f(x) = (x^2 + x + 1)q_1(x)$. Similarly, $g(x) = (x^2 + x + 1)q_2(x)$ for some polynomial $q_2(x)$, and hence $(f + g)(x) = (x^2 + x + 1)(q_1(x) + q_2(x))$. This shows that the subset is closed under vector addition. Similarly, if $c$ is a scalar, $(cf)(x) = (x^2 + x + 1)(cq_1)(x)$, implying the $cf$ is divisible by $x^2 + x + 1$, so the subset is closed under scalar multiplication. The combination of the two properties demonstrates that the subset is a subspace.

15pts. VI. Compute the determinant of the matrices $E(n)$, where $E(n)$ is the matrix with 1’s down the main diagonal and 1’s on the diagonals to either side of the main diagonal, for $n = 2, 3, 4, 5$ and 6. For example, $E(5) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Explain what properties of determinants you used to answer this question.

The determinant of $E(2)$ is 0. The determinant of $E(3)$ is $-1$. After this, we can show that the determinant of $E(n)$ is the determinant of $E(n-1)$ minus the determinant of $E(n-2)$ for $n \geq 4$. This shows that the determinant of $E(4)$ is $-1$; the determinant of $E(5)$ is 0; and the determinant of $E(6)$ is 1. This pattern cycles every 6, so the determinant of $E(100)$ (the extra credit problem) is the same as the determinant of $E(4)$ which is $-1$. In order to see this relationship, expand the determinant about the first row: you will get the determinant of $E(n-1)$ minus the determinant of another similar $(n-1) \times (n-1)$ matrix. If you expand this second matrix about the first column, you will see that the determinant is equal to the determinant of $E(n-2)$ as claimed.

20pts. VII. Answer the following true/false questions. In each case, either give a brief (two complete sentences) justification for your answer or provide a counterexample.

A. If $B$ is an echelon form of a matrix $A$, then the pivot columns of $B$ form a basis for $Col A$.

False. Take the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. An echelon form for $A$ is $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and the pivot column of $B$ (the first column) is not even in the column space of $A$. 
B. If the columns of an $n \times n$ matrix are linearly independent, then the determinant of that matrix is 0.
   False. Take the matrix \[
   \begin{pmatrix}
   1 & 1 \\
   1 & 2
   \end{pmatrix}
   \]. The columns of this matrix are linearly independent, but the determinant is $1 \neq 0$.

C. A subset $H$ of a vector space $V$ is a subspace of $V$ if the zero vector is in $H$.
   False. Take the union of the $x$ and $y$-axes in $\mathbb{R}^2$. This subset contains the zero vector, but it is not a subspace since we can add $[1, 0]$ and $[0, 1]$ and get something outside the subset.

D. Any set of $n$ vectors in $\mathbb{R}^m$ must be linearly independent if $m > n$.
   False. Take the vectors $[1, 1, 1]$ and $[2, 2, 2]$ in $\mathbb{R}^3$: that gives us $n = 2$ linearly independent vectors in $\mathbb{R}^m$ for $m = 3$, contrary to the claim.

E. If $\{v_1, \ldots, v_p\}$ is a spanning set for a vector space $V$, then $\dim V \geq p$.
   False. Take the vectors $[1, 0], [0, 1], [1, 1]$ as a spanning set for $\mathbb{R}^2$. There are 3 vectors in this collection, but the dimension of $\mathbb{R}^2$ is 2.