Show all work; unjustified answers may receive less than full credit.

(20pts.) 1. a. Find the first four terms of the MacLaurin series for \( \ln(1 + x^2) \) (hint: start with the MacLaurin series for \( \frac{2x}{1 + x^2} \)).
   b. Use the answer from part a. to find \( \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{n^2} \right) \).

(20pts.) 2. a. Compute the first four nonzero terms of the Taylor series for the function \( f(x) = \ln(3x + 2) \) about \( a = 1 \).
   b. Use the degree \( n = 3 \) Taylor polynomial for \( f(x) \) about \( a = 1 \) to estimate \( f(4/3) = \ln(6) \), and use Taylor’s inequality to estimate the error for \( \frac{2}{3} \leq x \leq \frac{4}{3} \) for this Taylor polynomial.

(20pts.) 3. Estimate \( \int_0^1 e^{-x^2} \, dx \) within .1, and justify your answer.

(20pts.) 4. A 500 Liter tank contains salt water with a concentration of .01 kg/Liter of salt. You pour in new salt water with a concentration of .08 kg/Liter of salt at a rate of 20 Liters per minute. Assuming that the tank is instantaneously mixed and that you pour the mixture out at 20 Liters per minute, how long does it take for the tank to reach a concentration of .05 kg/Liter?

(20pts.) 5. A roast turkey is taken from an oven when the temperature has reached 180 degrees centigrade and placed on a table in a room where the temperature is 20 degrees centigrade.
   a. If the temperature \( T \) satisfies the differential equation \( \frac{dT}{dt} = k(T - 20) \), solve the differential equation for \( T \) and verify that your solution satisfies the differential equation.
   b. If \( k = -.03 \). use Euler’s method with step size \( h = 5 \) to estimate the temperature of the turkey after 10 minutes.
   c. If the temperature is 140 degrees after 10 minutes, how long does it take to cool to 100 degrees? (Don’t use the estimate for \( k \) from part b: you should compute that value from the information in the problem).

Have a great Thanksgiving!
1. \( \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \)

\( \frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \ldots \)

\( \ln(1+x^2) = \int \frac{2x}{1+x^2} \, dx = \int \left( 2x - 2x^3 + 2x^5 - 2x^7 + \ldots \right) \, dx \)

\[ = C + x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots \]

\( x = 0: \quad \ln(1) = C + 0 \quad \Rightarrow \quad C = 0 \)

(a) \( \ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots \)

(b) \( x = \frac{1}{2}: \quad \ln \left( 1 + \left( \frac{1}{2} \right)^2 \right) = \left( \frac{1}{2} \right)^2 - \frac{1}{2} \left( \frac{1}{2} \right)^4 + \frac{1}{3} \left( \frac{1}{2} \right)^6 - \frac{1}{4} \left( \frac{1}{2} \right)^8 + \ldots \)

\[ \boxed{0.223} = \ln \left( \frac{5}{4} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^{2n}} \]

2. \( f(x) = \ln(3x+2) \)

\( f'(x) = \frac{3}{3x+2} \)

\( f''(x) = -\frac{9}{(3x+2)^2} \)

\( f'''(x) = \frac{54}{(3x+2)^3} \)

\( f^{(4)}(x) = -\frac{162x}{(3x+2)^4} \)

\( M = \left| f^{(4)} \left( \frac{1}{3} \right) \right| = \frac{486}{256} \)

(a) \( \ln(3x+2) = \ln 5 + \frac{3}{5} (x-1) \)

\( -\frac{9}{50} (x-1)^2 + \frac{54}{750} (x-1)^3 + \ldots \)

(b) \( \ln(6) \approx \ln 5 + \frac{3}{5} (\frac{1}{3}) \)

\[ = \frac{486}{256} \left( \frac{1}{3} \right)^3 \approx 0.000977 \]

\[ R_3 \left( \frac{1}{3} \right) \leq \frac{486}{256} \left( \frac{1}{3} \right)^3 \left( \frac{1}{3} \right)^{3+1} = 0.000977 \]
3. \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \)

\[
\int_0^1 e^{-x^2} \, dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right) \, dx
\]

\[= x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots \bigg|_0^1\]

\[= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \ldots\]

Alternating series \(\Rightarrow \int_0^1 e^{-x^2} \, dx \approx 1 - \frac{1}{3} = \boxed{\frac{2}{3}}\)

\[|\text{error}| < \frac{1}{70} = .01\]

4. \(y = \text{amt of salt at time } t\)

\[
\frac{dy}{dt} = \text{rate in} - \text{rate out} = 0.8 \frac{kg}{L} \times 20 \frac{L}{min} - \frac{y}{500} \frac{kg}{L} \times 20 \frac{L}{min}
\]

\[= \frac{800}{500} - \frac{20y}{500} = \frac{40 - y}{25}\]

\[
\int \frac{dy}{40 - y} = \int \frac{1}{25} \, dt \Rightarrow - \ln |40 - y| = \frac{1}{25} t + C
\]

\[\Rightarrow 40 - y = Ce^{\frac{1}{25} t} \Rightarrow y = 40 - Ce^{-\frac{1}{25} t}\]

When \(t = 0\) \(y = (0.01) \frac{kg}{L} \times 500L = 5 \text{ kg} = 40 - C \Rightarrow C = 35\)

\(y = 40 - 35e^{-\frac{1}{25} t}\)

\(0.05 \frac{kg}{L} \text{ conc} \Rightarrow 500L \times 0.05 \frac{kg}{L} = 25 \text{ kg}\)

\(0.05 \frac{kg}{L}\)

\[25 = 40 - 35e^{-\frac{1}{25} t}\]

\[e^{-\frac{t}{25}} = \frac{15}{35}\]

\[t = -25 \ln \left(\frac{15}{35}\right) = 21.18 \text{ min}\]
5. (a) \[ \frac{dT}{dt} = K(T-20) \]

\[ \int \frac{dT}{T-20} = \int K \, dt \]

\[ \ln|T-20| = KT + C \Rightarrow T-20 = C'e^{kt} \]

\[ T = 20 + C'e^{kt} \]

Verification: \[ \frac{dT}{dt} = C'e^{kt}(k) = K(T-20) \]

(b) \( y_0 = 180 \)
\( y_1 = 180 + (-.03)(180-20)(5) = 180 - 24 = 156 \)
\( y_2 = 156 + (-.03)(156-20)(5) = 156 - 20.4 = 135.6 \)

(c) \[ 140 = 20 + 160e^{Kt(10)} \]

\[ e^{Kt(10)} = .75 \]

\[ K = \frac{\ln(.75)}{10} = -.029 \]

\[ 100 = 20 + 160e^{-0.029t} \]

\[ t = \frac{\ln(0.5)}{-0.029} = 24.1 \text{ minutes} \]