Quiz 5

Davis
M212

Name:
Pledge:

(8pts.) 1. Find a power series representation for the function \( f(x) = \frac{1}{(1+x)^2} \) and determine the interval of convergence.

If we integrate this, we get 
\[-\frac{1}{4x+1} = -\frac{1}{4} \left(1 - \frac{x}{4} + \frac{x^2}{4^2} - \frac{x^3}{4^3} + \cdots\right) = -\frac{1}{4} \sum_{i=0}^{\infty} (-1)^i (\frac{x}{4})^i\]

by geometric series techniques. To get the original function, we differentiate, giving 
\[ f(x) = \frac{1}{(4+x)^2} = \frac{1}{16} - \frac{x}{32} + \frac{3x^2}{256} - \frac{4x^3}{1024} + \cdots = \sum_{i=0}^{\infty} (-1)^i (\frac{1}{4})^{i+2}(i+1)x^i. \]

The radius of convergence of the geometric series of 4, and this doesn’t change when we differentiate, so this series will converge for \(-4 < x < 4\).

(8pts.) 2. Find four nonzero terms of the Taylor series for \( \sin(2x) \) about \( a = \frac{\pi}{6} \).

Set up the table:

<table>
<thead>
<tr>
<th>functions and derivatives</th>
<th>plug in ( a = \frac{\pi}{6} )</th>
<th>coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \sin(2x) )</td>
<td>( f(2 \cdot \frac{\pi}{6}) = \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
</tr>
<tr>
<td>( f'(x) = 2 \cos(2x) )</td>
<td>( f'(2 \cdot \frac{\pi}{6}) = \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( f''(x) = -4 \sin(2x) )</td>
<td>( f''(2 \cdot \frac{\pi}{6}) = -2 \frac{\sqrt{3}}{3} )</td>
<td>( -2 \frac{\sqrt{3}}{3} )</td>
</tr>
<tr>
<td>( f'''(x) = -8 \cos(2x) )</td>
<td>( f'''(2 \cdot \frac{\pi}{6}) = -4 )</td>
<td>( -4 )</td>
</tr>
</tbody>
</table>

Thus, the Taylor series is 
\[ \sin(2x) = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{6}) - \frac{\sqrt{3}}{3}(x - \frac{\pi}{6})^2 - \frac{2}{3}(x - \frac{\pi}{6})^3 + \cdots. \]

(4pts.) 3. Explain why \( c_3 = f'''(a)/3! \) in the Taylor series \( f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \).

If we take the third derivative of the series expansion of \( f(x) \), we get:
\[ f'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + 5 \cdot 4 \cdot 3c_5(x-a)^2 + \cdots \]

Plug in \( x = a \) and get:
\[ f'''(a) = 3 \cdot 2c_3 + 0 + 0 + \cdots \]

Solving this for \( c_3 \) gives \( c_3 = \frac{f'''(a)}{3!} \) as claimed.