Show all work; unjustified answers may receive less than full credit.

(20pts.) 1. Without using a calculator, compute $\frac{dy}{dx}$ of the following functions.
   a. $y = \cos^2(x^2 + 1)$
   b. $y = \sin^{-1}(x^{\frac{3}{2}})$
   c. $y = e^{x^2} \ln(x^2 + x + 1)$
   d. $x^3 - 3xy + y^4 = 5$

(16pts.) 2. Use implicit differentiation to show that $(\sin^{-1}(x))' = \frac{1}{\sqrt{1-x^2}}$.

(16pts.) 3. A manufacturer produces bolts of fabric with a fixed width. The quantity $q$ of this fabric (measured in yards) that is sold is a function of the selling price $p$ (in dollars per yard), so we can write $q = f(p)$.
   a. What does it mean to say $f(10) = 100,000$ and $f'(10) = -3500$? (Explain in words including yards of fabric, price, and quantity sold.)
   b. The total revenue earned with selling price $p$ is $R(p) = pf(p)$. Assuming the values in part (a), find $R'(10)$ and interpret your answer (again using words as in part a).

(16pts.) 4. Use a linear approximation to the function $y = x^e$ at the point $a = 1$ to estimate $(.97)^{.97}$.

(16pts.) 5. A street light is mounted at the top of a 20-foot-tall pole. A 4 foot tall child runs away from the pole with a speed of 10 feet per second along a straight path. How fast is the tip of her shadow moving when she is 30 feet from the pole?

(16pts.) 6. Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 6x^2 + 9x + 2$ on the interval $[-1, 4]$. 
Key

1. a. \[ y = (\cos(x^2 + 1))^5 \implies \frac{dy}{dx} = 5 \cos^4(x^2 + 1) \left[ -\sin(x^2 + 1) (2x) \right] \]

b. \[ \frac{dy}{dx} = \frac{1}{\sqrt{1-x^6}} \cdot \frac{3}{5} x^{-\frac{3}{5}} \]

c. \[ \frac{dy}{dx} = e^x \cdot \frac{2x+1}{x^2 + x + 1} + 2xe^x \ln(x^2 + x + 1) \]

d. \[ 2x - 3xy' - 3y + 4y^3 y' = 0 \]
\[ y'(4y^3 - 3x) = 3y - 2x \]
\[ y' = \frac{3y - 2x}{4y^3 - 3x} \]

2. \[ y = \sin^{-1}(x) \implies \sin(y) = x \implies \cos(y) y' = 1 \]
\[ \implies y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-(\sin(\sin^{-1}(x)))^2}} = \frac{1}{\sqrt{1-x^2}} \]

3. a. \( f(10) = 100,000 \) means we will sell 100,000 yards of fabric if the price is $10 per yard. \( f'(10) = -3500 \) means we will sell 3500 fewer yards of fabric for every dollar increase in price starting from $10 per yard.
3.6.  \[ R(p) = pf(p) \Rightarrow R'(p) = pf'(p) + f(p) \]
\[ \Rightarrow R'(10) = 10(-3500) + 100,000 = 65,000 \]
Our revenue will be increasing at a rate of $65,000 per $1 increase in price (selling fewer yards at a higher price per yard will yield more revenue.)

4. \[ y = x^x \Rightarrow \ln y = x \ln x \Rightarrow \frac{1}{y} y' = x \left( \frac{1}{x} \right) + \ln x \]
\[ \Rightarrow y' = (1 + \ln x)y = (1 + \ln x)x^x \]
\[ y'|_{x=1} = (1+\ln 1)1^1 = 1 = m_{\tan} \]

Tangent line: \[ y - 1 = 1(x - 1) \Rightarrow y = x \approx x^x \]
\[ (0.97)^{0.97} \approx 0.97 \]

5.

Use similar triangles:
\[ \frac{20}{4} = \frac{y}{y-x} \Rightarrow 5y - 5x = y \]
\[ \Rightarrow 4y = 5x \]
Take \( \frac{d}{dt} \) of both sides:
\[ 4 \frac{dy}{dt} = 5 \frac{dx}{dt} = 5(10) \Rightarrow \frac{dy}{dt} = 12.5 \text{ ft/s} \]
6. \[ f(x) = x^3 - 6x^2 + 9x + 2 \]

\[ f'(x) = 3x^2 - 12x + 9 = 0 \]

\[ 3(x-1)(x-3) = 0 \]

\[ x = 1, 3 \] critical points

\[ \begin{array}{c|c}
    x & f(x) \\
    \hline
    -1 & -14 \leftarrow \text{absolute min} \\
    1 & 6 \\
    3 & 2 \leftarrow \text{absolute max} \\
    4 & 6 \\
\end{array} \]