ABSOLUTE CONTINUITY OF A CLASS OF UNITARY OPERATORS

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Abstract. Let F denote the Fourier transform on $L^2(\mathbb{R})$, and let $T \equiv D_\varphi M_u$, where $D_\varphi \equiv F M_\varphi F^{-1}$, $\varphi \in H^\infty(\mathbb{R})$ is inner, and $|u| = 1$ a.e. This paper gives a partial description of the spectral multiplicity theory of T. It is shown that T is absolutely continuous and is a bilateral shift of infinite multiplicity if φ is not a finite Blaschke product. Similar results are obtained for the (isometric) restrictions of T to the invariant subspaces $L^2(\alpha, \infty)$. Specifically, these restrictions always have absolutely continuous unitary parts, and shift parts with multiplicity equal to the multiplicity of φ.

1. Introduction

Consider the class of operators on $L^2(\mathbb{R})$ consisting of sums and products of operators of the form $D_\varphi \equiv F M_\varphi F^{-1}$ and M_ψ, where F denotes the Fourier transform

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt} f(t)dt$$

and M_ψ denotes multiplication by ψ. For various choices of φ and ψ, one can obtain the Toeplitz, Hankel, and singular integral operators, as well as the convolution integral operators. The Fredholm theory of these objects has a long history, including studies by Duducava [6] and Power [16, 17, 18, 19], for φ, ψ both piecewise continuous. The more delicate problem of unitary equivalence has been solved only in the cases of self-adjoint Toeplitz operators [20, 11], self-adjoint singular integral operators [21, 15], and for certain classes of self-adjoint Hankel operators [8, 9, 10]. Most recently, operators of the form $D_\varphi M_\psi$ with φ and ψ

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analytic are featured in [12], in which the operators are used to determine the unitary automorphism group of the Fourier binest algebra.

This paper is the first in a series dealing with the class of unitary operators of the form $T \equiv D_\varphi M_u$, for $\varphi \in H^\infty(\mathbb{R})$ inner and $|u| = 1$ a.e. The present article focuses specifically on the issue of absolute continuity, and in the process answers the unitary equivalence question for the case in which φ is not a finite multiplicity Blashcke product. The next two papers in the series address the issue of unitary equivalence for the finite Blaschke product case. In each paper, ideas from spectral multiplicity theory are used to obtain information concerning the unitary equivalence classes of these objects, and also of the restrictions $T_{\alpha,\infty}$ of T to the invariant subspaces $L^2(\alpha, \infty)$.

Specifically, let $\mathbb{D}$ denote the open unit disk in the complex plane, and $\partial\mathbb{D}$ denote the unit circle. According to spectral theory, there is associated with each unitary operator U acting on a separable Hilbert space $\mathcal{H}$ a spectral measure E on $\partial\mathbb{D}$ such that

$$U = \int_{\partial\mathbb{D}} \lambda dE(\lambda).$$

U is *absolutely continuous* (resp. *singular*) if all of the scalar spectral measures $\langle E(\cdot)x, y\rangle$ are absolutely continuous (singular). In addition, there exist direct sum decompositions $\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_s$ and $U = U_{ac} \oplus U_s$ of $\mathcal{H}$ and U such that $\mathcal{H}_{ac}$ and $\mathcal{H}_s$ are U reducing subspaces, with $U_{ac} \equiv U|_{\mathcal{H}_{ac}}$ and $U_s \equiv U|_{\mathcal{H}_s}$ respectively absolutely continuous and singular [3, 13].

Spectral multiplicity theory provides a similar, but somewhat more powerful, formulation of these ideas. Specifically, it states that there is associated with U a Borel measurable field of Hilbert spaces $\{H_\lambda : \lambda \in \partial\mathbb{D}\}$ and a finite positive Borel measure ν on $\partial\mathbb{D}$ such that U is unitarily equivalent to multiplication by λ on the direct integral Hilbert space

$$\mathcal{D} = \int_{\partial\mathbb{D}} \oplus H_\lambda d\nu(\lambda).$$

The *spectral multiplicity function* n defined by $n(\lambda) = \dim H_\lambda$ along with the scalar spectral measure ν describe U in the sense that any unitary operator whose scalar spectral measure is mutually absolutely continuous with ν and whose multiplicity function agrees with n except on a set of ν measure zero must be unitarily equivalent to U. The properties of absolute continuity and singularity in this framework are defined to reflect the corresponding properties of ν. Despite the apparent differences between these definitions and those in the previous paragraph, both sets of definitions describe the same subclasses of unitary operators.

Section 2 of this note provides a brief introduction to Hardy spaces on the half-plane, along with other relevant definitions and concepts. Sections 3 and 4 address the absolute continuity problem. In the former, operators T for which φ has the form $\varphi = e^{i\alpha x}$, $\alpha > 0$ are examined with the help of the minimal unitary dilation notion of Sz.Nagy-Foiaş [23]. The final section presents an example of an operator that is not absolutely continuous, demonstrating that a "slightly" non-analytic φ can result in a non-trivial point spectrum.

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2. PRELIMINARIES

Let Ω denote the upper half-plane, and let H^2 denote the Hardy space on Ω, which by passing to boundary values may be considered a subspace of $L^2(\mathbb{R})$. The Paley-Wiener Theorem (see e.g. [22]) states that F maps $e^{i\alpha x}H^2$ isometrically onto $L^2(\alpha, \infty)$. One should note that if $\alpha < 0$, then $e^{i\alpha x}H^2$ contains H^2. The space H^∞ consists of all bounded analytic functions on the upper half-plane, and can be considered a subspace of $L^\infty(\mathbb{R})$ in a manner similar to H^2.

A function $\varphi \in H^\infty$ is said to be *inner* provided $|\varphi| = 1$ almost everywhere on $\mathbb{R}$. For such φ, the set $\varphi H^2 \equiv \{\varphi f : f \in H^2\}$ is a closed subspace of H^2. Inner functions of the form

$$B(z) = \left(\frac{z-i}{z+i}\right)^m \prod_n \frac{|z_n^2+1|}{z_n^2+1}\frac{z-z_n}{z-\overline{z_n}}$$

for m and n nonnegative integers and $\{z_n\}$ a sequence in $\Omega\backslash\{i\}$ with

$$\sum_n \frac{y_n}{1+|z_n|^2} < \infty, \quad z_n = x_n + iy_n$$

are called *Blaschke products*, and their *multiplicity* is defined to be the number of factors if this number is finite, and infinite if not. In general we define the multiplicity of an inner function to be the dimension of the subspace $H^2 \ominus \varphi H^2$. For Blaschke products these two definitions coincide. It can be shown that an inner function φ has finite multiplicity if and only if it is a finite multiplicity (or simply "finite") Blaschke product, and in this case, the elements of $H^2 \ominus \varphi H^2$ are all rational functions.

For $\varphi \in L^\infty(\mathbb{R})$ we define the *Toeplitz operator with symbol* φ, denoted T_φ, to be the operator on H^2 given by

$$T_\varphi f = PM_\varphi f = P(\varphi f),$$

where P denotes the orthogonal projection of $L^2(\mathbb{R})$ onto H^2.

For $\alpha \geq 0$, let S_α be the unitary translation operator on $L^2(\mathbb{R})$ defined by

$$(S_\alpha f)(x) = f(x-\alpha)$$

and let $\hat{S}_\alpha$ denote the isometric restriction to $L^2(0,\infty)$ defined by

$$(\hat{S}_\alpha f)(x) = \chi_{(\alpha,\infty)}(x) f(x-\alpha), \quad (\hat{S}_\alpha^* f)(x) = \chi_{(0,\infty)}(x) f(x+\alpha).$$

When considering F as a unitary mapping on $H^2(\mathbb{R})$, the relations

$$FT_{e^{i\alpha x}} = \hat{S}_\alpha F$$

and

$$\hat{S}_\alpha^* \hat{S}_\alpha = I, \quad \hat{S}_\alpha \hat{S}_\alpha^* = P_{\alpha,\infty}$$

are useful. Here, $P_{\alpha,\infty}$ denotes the projection of $L^2(0,\infty)$ onto $L^2(\alpha,\infty)$.

For T a contraction on a separable Hilbert space $\mathcal{H}$, define the *defect operators* D_T and D_{T^*}, and *defect spaces* $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$, by

$$D_T = (I - T^*T)^{\frac{1}{2}}, \quad D_{T^*} = (I - TT^*)^{\frac{1}{2}}$$

and

$$\mathcal{D}_T = \overline{D_T(\mathcal{H})}, \quad \mathcal{D}_{T^*} = \overline{D_{T^*}(\mathcal{H})},$$

where the bars denote closure in the norm topology of $\mathcal{H}$. It is easily shown that if T is isometric, then the defect operator D_{T^*} is the orthogonal projection of $\mathcal{H}$ onto the closed subspace range$(I - TT^*)$.

For the remainder of this paper, we assume that T is the unitary operator on $L^2(\mathbb{R})$ given by

$$T \equiv FM_\varphi F^{-1} M_u,$$

where $\varphi \in H^\infty$ is inner and $u \in L^\infty$ is such that $|u| = 1$ almost everywhere. When convenient, we will use D_φ to denote the Fourier multiplier $D_\varphi = FM_\varphi F^{-1}$ so that T may also be written $T = D_\varphi M_u$. Much of the analysis that follows benefits from the rich collection, $\{L^2(\alpha,\infty) : \alpha \in \mathbb{R}\}$, of T invariant subspaces of $L^2(\mathbb{R})$. To see this invariance, note that the Paley-Wiener Theorem combined with the relation

$$FM_{e^{i\alpha x}} = S_\alpha F$$

implies that F maps $e^{i\alpha x}H^2$ isometrically onto $L^2(\alpha,\infty)$. From this, it follows that

$$D_\varphi(L^2(\alpha,\infty))M_u = FM_\varphi(e^{i\alpha x}H^2) = F(\varphi e^{i\alpha x}H^2) \subseteq F(e^{i\alpha x}H^2) = L^2(\alpha,\infty).$$

Since the present analysis involves a large number of subspaces of various forms, some notational shortcuts are employed. For $-\infty \le \alpha < \beta \le \infty$, and A a bounded operator on a Hilbert space $\mathcal{H}$, we will write $A_{\alpha,\beta}$ to denote the compression of A to $L^2(\alpha,\beta)$, when this compression exists. The lone exceptions to this convention are that $\hat{T} = T|_{L^2(0,\infty)}$ and (as mentioned previously) $\hat{S}_\alpha = S_\alpha|_{L^2(\alpha,\infty)}$. $P_{\mathcal{M}}$ denotes the orthogonal projection of $\mathcal{H}$ onto the subspace $\mathcal{M}$, and $\mathcal{M}^\perp$ refers to the subspace $\mathcal{H}\ominus\mathcal{M}$. In many of the examples that follow, $\mathcal{M}$ will be a subspace of both L^2 and H^2, in which case $\mathcal{M}^\perp$ will always refer to $H^2\ominus\mathcal{M}$ as opposed to $L^2\ominus\mathcal{M}$. If A is a contraction, we will use A_u, A_0; H_u, $\mathcal{H}_0$; and P_u, P_0; to denote respectively the unitary and completely non-unitary parts of A, the corresponding subspaces of $\mathcal{H}$, and the projections onto these subspaces, assuming that such objects exist for the particular operator in question.

The analysis here is simplified by the unitary equivalence of several of the operators under consideration. In particular, for f in the L^2-dense set $L^1(0,\infty)\cap L^2(\alpha,\infty)$, and $\alpha\in\mathbb{R}$,

$$\begin{aligned}
S_\alpha^* P_{\alpha,\infty} D_\varphi M_u P_{\alpha,\infty} &= S_\alpha^* P_{\alpha,\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\varphi(t)\int_\alpha^\infty e^{its}u(s)f(s)\,ds\,dt\right)\\
&= P_{0,\infty} S_\alpha^*\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\varphi(t)\int_\alpha^\infty e^{its}u(s)f(s)\,ds\,dt\right)\\
&= P_{0,\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\varphi(t)\int_\alpha^\infty e^{i(s-\alpha)t}u(s)f(s)\,ds\,dt\right)\\
&= P_{0,\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\varphi(t)\int_0^\infty e^{its}u(s+\alpha)f(s+\alpha)\,ds\,dt\right)\\
&= P_{0,\infty} D_\varphi M_{S_\alpha{}^* u} P_{0,\infty} S_\alpha^*,
\end{aligned}$$

so that for all $f\in L^2(\alpha,\infty)$,

$$(D_\varphi M_u)_{\alpha,\infty} = S_\alpha (D_\varphi M_{S_\alpha{}^* u})_{0,\infty} S_\alpha^*. \tag{2.1}$$

This unitary equivalence allows us to concentrate our isometric case efforts on the $L^2(0,\infty)$ restriction.

Finally, we assume throughout that φ is not a constant function, as this would make T a constant multiple of a multiplication operator, a class whose spectral multiplicity theory is well understood [1].

3. ABSOLUTE CONTINUITY FOR $\varphi = e^{i\alpha x}$, $\alpha > 0$.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and suppose $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. We write $A = prB$, if both $\mathcal{H}$ is a subspace of $\mathcal{K}$ and $Ax = P_{\mathcal{H}}Bx$ for all $x \in \mathcal{H}$, where $P_{\mathcal{H}}$ denotes the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. B is a *dilation* of A if $A^n = prB^n$ for $n = 1, 2, \ldots$. If in addition B is unitary and $\mathcal{K}$ satisfies

$$\mathcal{K} = \bigvee_{n=-\infty}^{\infty} B^n \mathcal{H},$$

then B is the *minimal unitary dilation* of A. (For a given operator A, there can be many different dilations satisfying this definition. All of these turn out to be isomorphic, however, and thus we speak of "the" minimal unitary dilation.)

Now, let $\varphi = e^{i\alpha x}$ for some $\alpha > 0$. Then

$$T = FM_{e^{i\alpha x}}F^{-1}M_u = S_\alpha FF^{-1}M_u = S_\alpha M_u$$

so

$$\begin{aligned} \hat{T}^* = P_{0,\infty}T^*|_{L^2(0,\infty)} &= P_{0,\infty}M_{\overline{u}}S_\alpha^*|_{L^2(0,\infty)} = \\ &= M_{\overline{u}}P_{0,\infty}S_\alpha^*|_{L^2(0,\infty)} = M_{\overline{u}}\S_\alpha{}^*|_{L^2(0,\infty)}. \end{aligned}$$

A direct calculation shows that for $n = 1, 2, \ldots$, and $f \in L^2(0,\infty)$,

$$(\hat{T}^{*n}f)(x) = \chi_{(0,\infty)}(x)f(x+n\alpha)\prod_{k=0}^{n-1}\overline{u(x+k\alpha)}. \tag{3.1}$$

Lemma 3.1. *The operator $T_{\beta,\infty}$ with φ as above is a unilateral shift of infinite multiplicity for all $\beta \in \mathbb{R}$.*

Proof. By equation (2.1), we may restrict our attention to $\hat{T}$. This is isometric, so by the Wold decomposition it suffices to show that $\hat{T}^{*n} \to 0$ strongly, and that $\dim(L^2(0,\infty) \ominus \hat{T}L^2(0,\infty)) = \infty$. The former follows easily from (3.1) and the latter from the relation $L^2(0,\infty) \ominus \hat{T}L^2(0,\infty) = L^2(0,\alpha)$. ☐

Lemma 3.2. *T is the minimal unitary dilation of $\hat{T}$.*

Proof. For $n = 1, 2, \ldots$ we have

$$\begin{aligned} \hat{T}^n &= \underbrace{(P_{0,\infty}T|_{L^2(0,\infty)})(P_{0,\infty}T|_{L^2(0,\infty)})\cdots\cdots(P_{0,\infty}T|_{L^2(0,\infty)})}_{n} \\ &= P_{0,\infty}T^n|_{L^2(0,\infty)} \end{aligned}$$

so $\hat{T}^n f = P_{0,\infty} T^n f$ for all $f \in L^2(0,\infty)$, and T is a unitary dilation of $\hat{T}$. Also,

$$\bigvee_{n=-\infty}^{\infty} T^n L^2(0,\infty) = \bigvee_{n=-\infty}^{\infty} L^2(n\alpha,\infty) = L^2(\mathbb{R})$$

so T is minimal. $\square$

Corollary 3.3. *For $\varphi = e^{i\alpha x}$ with $\alpha > 0$, T is a bilateral shift of infinite multiplicity, and thus absolutely continuous.*

4. ABSOLUTE CONTINUITY FOR THE REMAINING INNER FUNCTIONS

The following lemma is fundamental:

Lemma 4.1. *Let A be a contraction on $\mathcal{H}$, and let $\mathcal{L}$ be a collection of A invariant subspaces. For $\mathcal{M} \in \mathcal{L}$, let $A_{\mathcal{M}} = A|_{\mathcal{M}}$. If*

$$\bigvee \{P_{\mathcal{N}} \mathcal{D}_{A^*_{\mathcal{M}}} : \mathcal{M}, \mathcal{N} \in \mathcal{L}, \mathcal{N} \subseteq \mathcal{M}\} = \mathcal{H},$$

then A_u, the unitary part of A, is absolutely continuous.

Proof. Let $\mathcal{N} \subseteq \mathcal{M}$. We first show that

$$P_{\mathcal{N}} D^2_{A^*_{\mathcal{M}}} P_{\mathcal{N}} \le D^2_{A^*_{\mathcal{N}}}, \tag{4.2}$$

from which it follows by a "folk theorem" (see e.g. [4]) that

$$\operatorname{range}(P_{\mathcal{N}} D_{A^*_{\mathcal{M}}}) \subseteq \operatorname{range}(D_{A^*_{\mathcal{N}}}). \tag{4.3}$$

Toward this end, consider the matrix decomposition of $A_{\mathcal{M}}$ with respect to the direct sum $\mathcal{M} = \mathcal{M} \oplus (\mathcal{M} \ominus \mathcal{N})$. Since clearly $\mathcal{N}$ is invariant for $A_{\mathcal{M}}$, and $A_{\mathcal{M}}$ restricted to $\mathcal{N}$ is $A_{\mathcal{N}}$, this matrix has the form

$$A_{\mathcal{M}} = \begin{bmatrix} A_{\mathcal{N}} & X \\ 0 & Y \end{bmatrix}.$$

It follows that

$$\begin{aligned} P_{\mathcal{N}} D^2_{A^*_{\mathcal{M}}} P_{\mathcal{N}} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A_{\mathcal{N}} & X \\ 0 & Y \end{bmatrix} \begin{bmatrix} A^*_{\mathcal{N}} & 0 \\ X^* & Y^* \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I - A_{\mathcal{N}} A^*_{\mathcal{N}} - XX^* & 0 \\ -YX^* & I - YY^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I - A_{\mathcal{N}} A^*_{\mathcal{N}} - XX^* & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} D^2_{A^*_{\mathcal{N}}} - XX^* & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus, $P_{\mathcal{N}} D^2_{A^*_{\mathcal{M}}} P_{\mathcal{N}} = D^2_{A^*_{\mathcal{N}}} - XX^*$ on $\mathcal{N}$. Since both $P_{\mathcal{N}} D^2_{A^*_{\mathcal{M}}} P_{\mathcal{N}}$ and $D^2_{A^*_{\mathcal{N}}}$ are zero on $\mathcal{N}^\perp$, (4.2) holds, as does the range inclusion (4.3). Now, let $f \in P_{\mathcal{N}} D_{A^*_{\mathcal{M}}}(\mathcal{H})$. Then by (4.3), $f = D_{A^*_{\mathcal{N}}} h$ for some h, so

$$(1 - |\lambda|^2) \|(I - \lambda A)^{-1} f\|^2 = (1 - |\lambda|^2) \|(I - \lambda A_{\mathcal{N}})^{-1} \mathcal{D}_{A^*_{\mathcal{N}}} h\|^2 \le \|h\|^2.$$

It follows that if P_u is the projection of $\mathcal{H}$ onto H_u, the unitary subspace of A, then

$$(1 - |\lambda|^2) \|(I - \lambda A_u)^{-1} P_u f\|^2 \le \|h\|^2, \text{ all } \lambda \in \mathbb{D}.$$

In particular, the quantity on the left is bounded for all $\lambda \in \mathbb{D}$. By the spectral theorem, therefore, $P_u f$ is in the absolutely continuous subspace of H_u. But the vectors f as above span $\mathcal{H}$, so $\{P_u f : f \text{ as above}\}$ spans H_u and it follows that the absolutely continuous subspace of H_u is all of H_u. □

The plan is to apply this result to the operators $T_{\alpha,\infty}$, and then use the absolute continuity of $(T_{\alpha,\infty})_u$ to show the property for T. So, let us consider the operator $\hat{T}$ with φ neither constant nor of the form $e^{i\alpha x}$, and let $\mathcal{L}$ be the collection of invariant subspaces

$$\mathcal{L} = \{L^2(\alpha, \infty) : \alpha \ge 0\}.$$

We verify the stronger spanning condition

$$\bigvee \{P_{\mathcal{M}} \mathcal{D}_{\hat{T}^*} : \mathcal{M} \in \mathcal{L}\} = L^2(0, \infty).$$

Since $\hat{T}$ is isometric, $\mathcal{D}_{\hat{T}^*} = L^2(0,\infty) \ominus \hat{T} L^2(0,\infty)$, which in this particular case is simply $F(H^2 \ominus \varphi H^2)$, so we need only show that

$$\bigvee \{\chi_{(\alpha,\infty)} F(H^2 \ominus \varphi H^2) : \alpha \ge 0\} = L^2(0, \infty)$$

Toward this end, we prove the following:

Lemma 4.2. *Let φ be inner and non-constant. Then all of the elements of $F(H^2 \ominus \varphi H^2)$ vanish simultaneously on a set of positive measure if and only if $\varphi = e^{i\alpha x}$ for some $\alpha > 0$.*

Proof. The Paley-Wiener theorem renders one direction trivial. So, suppose there exists a set $\Delta \subseteq (0,\infty)$ with positive Lebesgue measure (denoted $|\Delta| > 0$), on which every f in $F(H^2 \ominus \varphi H^2)$ vanishes almost everywhere. Then, $F(H^2 \ominus \varphi H^2)$ is contained in $L^2((0,\infty)\backslash\Delta)$ and it follows that $(F^{-1}L^2((0,\infty)\backslash\Delta))^{\perp}$ is contained in φH^2. Since $F^{-1}L^2(\Delta)$ is orthogonal to $F^{-1}L^2((0,\infty)\backslash\Delta)$, it must also be contained in φH^2, and equivalently $L^2(\Delta) \subseteq F(\varphi H^2)$. It follows that for all $\alpha > 0$,

$$\hat{S}_\alpha L^2(\Delta) \subseteq \hat{S}_\alpha F(\varphi H^2) = FT_{e^{i\alpha x}}(\varphi H^2) \subseteq F(\varphi H^2),$$

and hence

$$\bigvee_{\alpha \in (0,\infty)} \hat{S}_\alpha L^2(\Delta) \subseteq F(\varphi H^2).$$

Let $\gamma = \inf\{t \in (0,\infty) : |\Delta \cap (0,t)| > 0\}$. One can show that the span on the left is exactly $L^2(\gamma,\infty)$ so that the latter subspace is contained in $F(\varphi H^2)$ and $e^{i\gamma x}H^2$ is contained in φH^2. Now γ can't be zero, since then H^2 would be contained in φH^2 and φ would be constant, a case we have excluded. If $\gamma > 0$, then the non-constant φ must divide $e^{i\gamma x}$. By standard function theory [7], $\varphi = e^{i\alpha x}$ for some α with $0 < \alpha \leq \gamma$. □

Proposition 4.3. *Let φ be an inner function which is neither a constant nor of the form $e^{i\alpha x}$, and let $T = D_\varphi M_u$ with $|u| = 1$ a.e. Then for all $\alpha \in \mathbb{R}$, the isometric restriction $T_{\alpha,\infty} \equiv T|_{L^2(\alpha,\infty)}$ has absolutely continuous unitary part, and shift part with multiplicity equal to the multiplicity of φ.*

Proof. We first consider the absolute continuity of $\hat{T}_u$. By Lemma 4.1 and the remarks following it, it suffices to show that

$$\bigvee\{\chi_{(\alpha,\infty)}F(H^2 \ominus \varphi H^2) : \alpha \geq 0\} = L^2(0,\infty).$$

Let $\mathcal{M}$ denote the span on the left and suppose $g \in L^2(0,\infty) \ominus \mathcal{M}$. Then the function $h(t)$ defined by

$$h(t) = -\int_\infty^t g(x)\overline{f(x)}dx$$

is zero for all $t \geq 0$ and all $f \in \mathcal{M}$. But h is absolutely continuous, so differentiating gives $g(x)\overline{f(x)} = 0$ almost everywhere. Now, let $E = \{x : g(x) \neq 0\}$. Clearly, $f(x) = 0$ almost everywhere on E for all $f \in F(H^2 \ominus \varphi H^2)$. By the previous lemma, $|E| = 0$, which implies that $g = 0$ almost everywhere, from which the absolute continuity of $\hat{T}_u$ follows. That the shift part of $\hat{T}$ has the stated multiplicity is clear. The result for the remaining $T_{\alpha,\infty}$ follows from equation (2.1). □

Theorem 4.4. Let $T = D_\varphi M_u$ with φ and u as above. Then T is absolutely continuous.

Proof. Fix $\alpha \in \mathbb{R}$ and $f \neq 0 \in L^2(\alpha, \infty)$. Let E and E_α denote the spectral measures for T and $(T_{\alpha,\infty})_u$ respectively, and let $(P_\alpha)_0$ and $(P_\alpha)_u$ denote the projections onto the completely non-unitary and unitary subspaces of $L^2(\alpha, \infty)$ with respect to the operator $T_{\alpha,\infty}$. For $\lambda \in \mathbb{D}$, and $\theta \in [0, 2\pi]$, define the Poisson kernel $P_\lambda(\theta)$ by

$$P_\lambda(\theta) = \frac{1 - |\lambda|^2}{|1 - \overline{\lambda} e^{i\theta}|^2}.$$

By the spectral theorem and the definition of $T_{\alpha,\infty}$, we have for all $\lambda \in \mathbb{D}$,

$$\begin{aligned}
\int_{\partial\mathbb{D}} P_\lambda(\theta) d\langle E(\theta) f, f\rangle &= (1 - |\lambda|^2) \|(I - \overline{\lambda} T)^{-1} f\|^2 \\
&= (1 - |\lambda|^2) \|(I - \overline{\lambda} T_{\alpha,\infty})^{-1} f\|^2 \\
&= (1 - |\lambda|^2) \|(I - \overline{\lambda} (T_{\alpha,\infty})_u)^{-1} (P_\alpha)_u f\|^2 \\
&\quad + (1 - |\lambda|^2) \|(I - \overline{\lambda} (T_{\alpha,\infty})_0)^{-1} (P_\alpha)_0 f\|^2 \\
&= \int_{\partial\mathbb{D}} P_\lambda(\theta) d\langle E_\alpha(\theta) (P_\alpha)_u f, (P_\alpha)_u f\rangle \\
&\quad + (1 - |\lambda|^2) \|(I - \overline{\lambda} (T_{\alpha,\infty})_0)^{-1} (P_\alpha)_0 f\|^2. \qquad (4.5)
\end{aligned}$$

Now, $(T_{\alpha,\infty})_0$ is a shift with multiplicity equal to the dimension of $\mathcal{D}_{\hat{T}^*}$, and hence unitarily equivalent to the shift operator $S : f(\lambda) \to \lambda f(\lambda)$ acting on the vector valued Hardy space $H^2(\mathcal{D}_{\hat{T}^*})$. It follows that there exists a non-zero $g \in H^2(\mathcal{D}_{\hat{T}^*})$ with

$$\|(I - \overline{\lambda} (T_{\alpha,\infty})_0)^{-1} (P_\alpha)_0 f\| = \|(I - \overline{\lambda} S)^{-1} g\|.$$

But, for $w \in \mathbb{D}$,

$$((I - \overline{\lambda} S)^{-1} g)(w) = \frac{1}{1 - \overline{\lambda} w} g(w),$$

so that

$$\begin{aligned}\left\|(I-\overline{\lambda}S)^{-1}g\right\|^2 &= \left\|\frac{1}{1-\overline{\lambda}w}g(w)\right\|^2 \\ &= \int_{\partial\mathbb{D}} \frac{1}{\left|1-\overline{\lambda}e^{i\theta}\right|^2}\left\|g(e^{i\theta})\right\|^2\frac{d\theta}{2\pi} \\ &= \frac{1}{1-|\lambda|^2}\int_{\partial\mathbb{D}} P_\lambda(\theta)\|g(e^{i\theta})\|^2\frac{d\theta}{2\pi},\end{aligned}$$

and hence

$$\left\|(I-\overline{\lambda}(T_{\alpha,\infty})_0)^{-1}(P_\alpha)_0 f\right\|^2 = \frac{1}{1-|\lambda|^2}\int_{\partial\mathbb{D}} P_\lambda(\theta)\|g(e^{i\theta})\|^2\frac{d\theta}{2\pi}.$$

Combining this with (4.5) gives

$$\begin{aligned}\int_{\partial\mathbb{D}} P_\lambda(\theta)d\langle E(\theta)f,f\rangle &= \int_{\partial\mathbb{D}} P_\lambda(\theta)d\langle E_\alpha(\theta)(P_\alpha)_u f,(P_\alpha)_u f\rangle \\ &\quad + \int_{\partial\mathbb{D}} P_\lambda(\theta)\|g(e^{i\theta})\|^2\frac{d\theta}{2\pi},\end{aligned}$$

all $\lambda\in\mathbb{D}$, from which it follows that

$$d\langle E(\theta)f,f\rangle = d\langle E_\alpha(\theta)(P_\alpha)_u f,(P_\alpha)_u f\rangle + \|g(e^{i\theta})\|^2\frac{d\theta}{2\pi}.$$

By Proposition 4.3 then, $d\langle E(\theta)f,f\rangle$ is absolutely continuous. Since $\alpha\in\mathbb{R}$ was arbitrary and $\bigvee_{\alpha\in\mathbb{R}} L^2(\alpha,\infty)=L^2(\mathbb{R})$, the result follows. $\square$

Corollary 4.5. *Let T be as above and suppose φ has infinite multiplicity. Then T is a bilateral shift of infinite multiplicity.*

Proof. For a bilateral shift U of infinite multiplicity, the scalar spectral measure is Lebesgue measure on the unit circle and the spectral multiplicity function is infinite almost everywhere. Since the direct sum of two unitary operators has scalar spectral measure and spectral multiplicity the sum of the corresponding elements of the summands, the direct sum of an absolutely continuous unitary and a bilateral shift of infinite multiplicity is a bilateral shift of infinite multiplicity.

Now, T is a unitary dilation of $\hat{T}$, so it must contain the minimal unitary dilation $\mathcal{U}$ of $\hat{T}$. Since the minimal unitary dilation of a unilateral shift is a bilateral shift of the same multiplicity, Proposition 4.3 implies that $\mathcal{U}$ is the direct sum of an absolutely continuous unitary operator and a bilateral shift of infinite multiplicity, which from the preceding paragraph is just a bilateral shift of infinite multiplicity. Now, let $\mathcal{K}\equiv\bigvee_{n=-\infty}^{\infty}T^nL^2(\alpha,\infty)$. Then $\mathcal{K}$ is reducing for T and by the definition of the minimal unitary dilation, $T|_{\mathcal{K}}$ is $\mathcal{U}$. Thus, $T = V \oplus \mathcal{U}$, where V is the restriction of T to $L^2(\mathbb{R}) \ominus \mathcal{K}$. By Theorem 4.4, T is absolutely continuous, which together with the absolute continuity of $\mathcal{U}$ implies that V is absolutely continuous as well. That is, T is the direct sum of an absolutely continuous unitary and a bilateral shift of infinite multiplicity, and is thus a bilateral shift of infinite multiplicity. □

5. A "COUNTEREXAMPLE"

The present example demonstrates that our conditions on the symbol functions φ and u are "sharp" as far as absolute continuity is concerned. Specifically, there exists a function φ, a quotient of two Blaschke factors, and a unimodular function u, such that $D_\varphi M_u$ is not absolutely continuous. I am grateful to J. Howland and T. Kriete for bringing it to my attention.

For $n \geq 0$, define the Hermite polynomials H_n and Hermite functions e_n by [2]

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$e_n(x) = H_n(x) e^{-\frac{x^2}{2}}.$$

Arguments from classical analysis [5] show that the e_n are eigenfunctions for the Fourier transform satisfying

$$Fe_n = (-i)^n e_n.$$

Let $f = e_0 + ie_2$, and $g = e_0 - ie_2$. Then

$$F^{-1} f = F^{-1}(e_0 + ie_2) = e_0 - ie_2 = g,$$

and similarly $F^{-1} g = f$. Moreover, since the e_i are real valued,

$$|f|^2 = {e_0}^2 + {e_2}^2 = |g|^2,$$

so that

$$|F^{-1} f|^2 = |g|^2 = |f|^2 = |F^{-1} g|^2$$

almost everywhere on $\mathbb{R}$. If we let $k = -\frac{\frac{3}{2} - \frac{i}{4}}{\frac{3}{2} + \frac{i}{4}}$, and $u = \frac{kg}{f}$, then u is unimodular, and since f is not a constant multiple of g, u is not constant. Now, let

$$\varphi = \frac{F^{-1} f}{kF^{-1} g}.$$

Then $|\varphi| = 1$ almost everywhere, and we have

$$F^{-1} f = \varphi k F^{-1} g = \varphi F^{-1}(uf) = M_\varphi F^{-1} M_u f$$

so that f is an eigenvector of $D_\varphi M_u$, and thus the latter has point spectra. But,

$$\varphi = \frac{F^{-1}f}{kF^{-1}g} = \frac{g}{kf} = \frac{e_0 - ie_2}{k(e_0 + ie_2)} = \frac{1 - iH_2}{k(1 + iH_2)}.$$

If for $z_0 \in \Omega$ we let $B_{z_0}(z)$ denote the single Blaschke factor with zero z_0, then a direct calculation shows that this last quotient is the boundary value of the complex function $\dfrac{B_\alpha(z)}{B_\beta(z)}$, where α is the upper half-plane square root of $\frac{1}{2} - \frac{i}{4}$ and $\beta = -\overline{\alpha}$. Since this last expression is a quotient of single Blaschke factors, φ is not inner, but is in many ways as close to being inner as any non-H^∞ function can be.

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