Solutions to the Sixtieth William Lowell Putnam Mathematical Competition

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A–1 Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r, s) = (r + s + |r - s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x - 3, 0\} - \max\{5x, 0\} + 3x + 2 \\ &= (-3x - 3 + |3x - 3|)/2 - (5x + |5x|)/2 + 3x + 2 \\ &= |(3x - 3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set $f(x) = (3x - 3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

A–2 First factor $p(x) = q(x)r(x)$, where q has all real roots and r has all complex roots. Notice that each root of q has even multiplicity, otherwise p would have a sign change at that root. Thus $q(x)$ has a square root $s(x)$.

Now write $r(x) = \prod_{j=1}^{k}(x - a_j)(x - \overline{a_j})$ (possible because r has roots in complex conjugate pairs). Write $\prod_{j=1}^{k}(x - a_j) = t(x) + iu(x)$ with t, x having real coefficients. Then for x real,

$$p(x) = q(x)r(x) = s(x)^2(t(x) + iu(x))(\overline{t(x) + iu(x)}) = (s(x)t(x))^2 + (s(x)u(x))^2.$$

A–3 First solution: Computing the coefficient of x^{n+1} in the identity $(1-2x-x^2)\sum_{m=0}^{\infty} a_m x^m = 1$ yields the recurrence $a_{n+1} = 2a_n + a_{n-1}$; the sequence $\{a_n\}$ is then characterized by this recurrence and the initial conditions $a_0 = 1, a_1 = 2$.

Define the sequence $\{b_n\}$ by $b_{2n} = a_{n-1}^2 + a_n^2,\ b_{2n+1} = a_n(a_{n-1} + a_{n+1})$. Then

$$\begin{aligned} 2b_{2n+1} + b_{2n} &= 2a_n a_{n+1} + 2a_{n-1}a_n + a_{n-1}^2 + a_n^2 \\ &= 2a_n a_{n+1} + a_{n-1}a_{n+1} + a_n^2 \\ &= a_{n+1}^2 + a_n^2 = b_{2n+2}, \end{aligned}$$

and similarly $2b_{2n} + b_{2n-1} = b_{2n+1}$, so that $\{b_n\}$ satisfies the same recurrence as $\{a_n\}$. Since further $b_0 = 1, b_1 = 2$ (where we use the recurrence for $\{a_n\}$ to calculate $a_{-1} = 0$), we deduce that $b_n = a_n$ for all n. In particular, $a_n^2 + a_{n+1}^2 = b_{2n+2} = a_{2n+2}$.

Second solution: Note that

$$\begin{aligned} \frac{1}{1 - 2x - x^2} &= \frac{1}{2\sqrt{2}}\left(\frac{\sqrt{2}+1}{1 - (\sqrt{2}+1)x} + \frac{\sqrt{2}-1}{1 - (1-\sqrt{2})x}\right) \\ &= \frac{1}{2\sqrt{2}}\left(\sum_{n=0}^{\infty}(\sqrt{2}+1)^{n+1}x^n - \sum_{n=0}^{\infty}(1-\sqrt{2})^{n+1}x^n\right), \end{aligned}$$

so that $a_n = \frac{1}{2\sqrt{2}}\left((\sqrt{2}+1)^{n+1} - (1-\sqrt{2})^{n+1}\right)$. A simple computation (omitted here) now shows that $a_n^2 + a_{n+1}^2 = a_{2n+2}$.

A–4 Denote the series by S, and note that

$$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(3^m/m)(3^m/m + 3^n/n)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(3^n/n)(3^m/m + 3^n/n)},$$

where the second equality follows by interchanging m and n. Thus

$$\begin{aligned} 2S &= \sum_m \sum_n \left(\frac{1}{(3^m/m)(3^m/m + 3^n/n)} + \frac{1}{(3^n/n)(3^m/m + 3^n/n)} \right) \\ &= \sum_m \sum_n \frac{1}{(3^m/m)(3^n/n)} \\ &= \left(\sum_{n=1}^{\infty} \frac{n}{3^n} \right)^2 . \end{aligned}$$

But $\sum_{n=1}^{\infty} n/3^n = 3/4$ (since, e.g., it's $f'(1)$, where $f(x) = \sum_{n=0}^{\infty} x^n/3^n = 3/(3-x)$), and we conclude that $S = 9/32$.

A–5 First solution: (by Reid Barton) Let $r_1, \dots, r_{1999}$ be the roots of P. Draw a disc of radius ϵ around each r_i, where $\epsilon < 1/3998$; this disc covers a subinterval of $[-1/2, 1/2]$ of length at most 2ϵ, and so of the 2000 (or fewer) uncovered intervals in $[-1/2, 1/2]$, one, which we call I, has length at least $\delta = (1 - 3998\epsilon)/2000 > 0$. We will exhibit an explicit lower bound for the integral of $|P(x)|/P(0)$ over this interval, which will yield such a bound for the entire integral.

Note that

$$\frac{|P(x)|}{|P(0)|} = \prod_{i=1}^{1999} \frac{|x - r_i|}{|r_i|}.$$

Also note that by construction, $|x - r_i| \geq \epsilon$ for each $x \in I$. If $|r_i| \leq 1$, then we have $\frac{|x-r_i|}{|r_i|} \geq \epsilon$. If $|r_i| > 1$, then

$$\frac{|x - r_i|}{|r_i|} = |1 - x/r_i| \geq 1 - |x/r_i| \geq 1 - 1/2 = 1/2 > \epsilon.$$

We conclude that $\int_I |P(x)/P(0)|\,dx \geq \delta\epsilon$, independent of P.

Second solution: It will be a bit more convenient to assume $P(0) = 1$ (which we may achieve by rescaling unless $P(0) = 0$, in which case there is nothing to prove) and to prove that there exists $D > 0$ such that $\int_{-1}^{1} |P(x)|\,dx \geq D$, or even such that $\int_0^1 |P(x)|\,dx \geq D$.

We first reduce to the case where P has all of its roots in $[0, 1]$. If this is not the case, we can factor $P(x)$ as $Q(x)R(x)$, where Q has all roots in the interval and R has none.

Then R is either always positive or always negative on $[0,1]$; assume the former. Let k be the largest positive real number such that $R(x) - kx \geq 0$ on $[0,1]$; then

$$\int_{-1}^{1} |P(x)|\,dx = \int_{-1}^{1} |Q(x)R(x)|\,dx > \int_{-1}^{1} |Q(x)(R(x) - kx)|\,dx,$$

and $Q(x)(R(x) - kx)$ has more roots in $[0,1]$ than does P (and has the same value at 0). Repeating this argument shows that $\int_0^1 |P(x)|\,dx$ is greater than the corresponding integral for some polynomial with all of its roots in $[0,1]$.

Under this assumption, we have $P(x) = c\prod_{i=1}^{1999}(x - r_i)$ for some $r_i \in (0,1]$. Since $P(0) = -c\prod r_i = 1$, we have $|c| \geq \prod |r_i^{-1}| \geq 1$.

Thus it suffices to prove that if $Q(x)$ is a *monic* polynomial of degree 1999 with all of its roots in $[0,1]$, then $\int_0^1 |Q(x)|\,dx \geq D$ for some constant $D > 0$. But the integral of $\int_0^1 \prod_{i=1}^{1999} |x - r_i|\,dx$ is a continuous function for $r_i \in [0,1]$. The product of all of these intervals is compact, so the integral achieves a minimum value for some r_i. This minimum is the desired D.

Note: combining the two approaches gives a constructive solution with a constant that is better, but is still far from optimal. I don't know offhand whether it is even known what the optimal constant and/or the polynomials achieving that constant are.

A–6 Rearranging the given equation yields the much more tractable equation

$$\frac{a_n}{a_{n-1}} = 6\,\frac{a_{n-1}}{a_{n-2}} - 8\,\frac{a_{n-2}}{a_{n-3}}.$$

Let $b_n = a_n/a_{n-1}$; with the initial conditions $b_2 = 2, b_3 = 12$, one easily obtains $b_n = 2^{n-1}(2^{n-2} - 1)$, and so

$$a_n = 2^{n(n-1)/2}\prod_{i=1}^{n-1}(2^i - 1).$$

To see that n divides a_n, factor n as $2^k m$, with m odd. Then note that $k \leq n \leq n(n-1)/2$, and that there exists $i \leq m - 1$ such that m divides $2^i - 1$, namely $i = \phi(m)$ (Euler's totient function: the number of integers in $\{1, \dots, m\}$ relatively prime to m).

B–1 The answer is $1/3$. Let G be the point obtained by reflecting C about the line AB. Since $\angle ADC = \frac{\pi-\theta}{2}$, we find that $\angle BDE = \pi - \theta - \angle ADC = \frac{\pi-\theta}{2} = \angle ADC = \pi - \angle BDC = \pi - \angle BDG$, so that E, D, G are collinear. Hence

$$|EF| = \frac{|BE|}{|BC|} = \frac{|BE|}{|BG|} = \frac{\sin(\theta/2)}{\sin(3\theta/2)},$$

where we have used the law of sines in $\triangle BDG$. But by l'Hôpital's Rule, $\lim_{\theta \to 0} \frac{\sin(\theta/2)}{\sin(3\theta/2)} = \lim_{\theta \to 0} \frac{\cos(\theta/2)}{3\cos(3\theta/2)} = 1/3$.

B–2 Suppose that P does not have n distinct roots; then it has a root of multiplicity at least 2, which we may assume is $x = 0$ without loss of generality. Let x^k be the greatest power of x dividing $P(x)$, so that $P(x) = x^k R(x)$ with $R(0) \neq 0$; a simple computation yields

$$P''(x) = k(k-1)x^{k-2}R(x) + 2kx^{k-1}R'(x) + x^k R''(x).$$

Since $R(0) \neq 0$ and $k \geq 2$, we conclude that the greatest power of x dividing $P''(x)$ is x^{k-2}. But $P(x) = Q(x)P''(x)$, and so x^2 divides $Q(x)$. We deduce (since Q is quadratic) that $Q(x)$ is a constant C times x^2; in fact, $C = 1/(n(n-1))$ by inspection of the leading-degree terms of $P(x)$ and $P''(x)$.

Now if $P(x) = \sum_{j=0}^{n} a_j x^j$, then the relation $P(x) = Cx^2P''(x)$ implies that $a_j = Cj(j-1)a_j$ for all j; hence $a_j = 0$ for $j \leq n-1$, and we conclude that $P(x) = a_n x^n$, which has all identical roots.

B–3 We first note that

$$\sum_{m,n>0} x^m y^n = \frac{xy}{(1-x)(1-y)}.$$

Subtracting S from this gives two sums, one of which is

$$\sum_{m \geq 2n+1} x^m y^n = \sum_n y^n \frac{x^{2n+1}}{1-x} = \frac{x^3 y}{(1-x)(1-x^2y)}$$

and the other of which sums to $xy^3/[(1-y)(1-xy^2)]$. Therefore

$$\begin{aligned} S(x,y) &= \frac{xy}{(1-x)(1-y)} - \frac{x^3y}{(1-x)(1-x^2y)} - \frac{xy^3}{(1-y)(1-xy^2)} \\ &= \frac{xy(1+x+y+xy-x^2y^2)}{(1-x^2y)(1-xy^2)} \end{aligned}$$

and the desired limit is $\lim_{(x,y)\to(1,1)} xy(1+x+y+xy-x^2y^2) = 3$.

B–4 We make repeated use of the following fact: if f is a differentiable function on all of $\mathbb{R}$, $\lim_{x\to-\infty} f(x) \geq 0$, and $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$. (Proof: if $f(y) < 0$ for some x, then $f(x) < f(y)$ for all $x < y$ since $f' > 0$, but then $\lim_{x\to-\infty} f(x) \leq f(y) < 0$.)

From the inequality $f'''(x) \leq f(x)$ we obtain

$$f''f'''(x) \leq f''(x)f(x) < f''(x)f(x) + f'(x)^2$$

since $f'(x)$ is positive. Applying the fact to the difference between the right and left sides, we get

$$\frac{1}{2}(f''(x))^2 < f(x)f'(x).$$

Adding $\frac{1}{2}f'(x)f'''(x)$ to both sides and again invoking the original bound $f'''(x) \leq f(x)$, we get

$$\frac{1}{2}[f'(x)f'''(x) + (f''(x))^2] < f(x)f'(x) + \frac{1}{2}f'(x)f'''(x) \leq \frac{3}{2}f(x)f'(x).$$

Applying the fact again, we get

$$\frac{1}{2}f'(x)f''(x) < \frac{3}{4}f(x)^2.$$

Multiplying both sides by $f'(x)$ and applying the fact once more, we get

$$\frac{1}{6}(f'(x))^3 < \frac{1}{4}f(x)^3.$$

From this we deduce $f'(x) < (3/2)^{1/3}f(x) < 2f(x)$, as desired.

Note: I don't know what the best constant is, except that it is not less than 1 (because $f(x) = e^x$ satisfies the given conditions).

B–5 We claim that the eigenvalues of A are 0 with multiplicity $n-2$, and $n/2$ and $-n/2$, each with multiplicity 1. To prove this claim, define vectors $v^{(m)}$, $0 \leq m \leq n-1$, componentwise by $(v^{(m)})_k = e^{ikm\theta}$, and note that the $v^{(m)}$ form a basis for $\mathbb{C}^n$. (If we arrange the $v^{(m)}$ into an $n \times n$ matrix, then the determinant of this matrix is a Vandermonde product which is nonzero.) Now note that

$$(Av^{(m)})_j = \sum_{k=1}^{n} \cos(j\theta + k\theta)e^{ikm\theta} = \frac{1}{2}\left(e^{ij\theta}\sum_{k=1}^{n} e^{ik(m+1)\theta} + e^{-ij\theta}\sum_{k=1}^{n} e^{ik(m-1)\theta}\right).$$

Since $\sum_{k=1}^{n} e^{ik\ell\theta} = 0$ for integer ℓ unless $n \,|\, \ell$, we conclude that $Av^{(m)} = 0$ for $m = 0$ or for $2 \leq m \leq n-1$. In addition, we find that $(Av^{(1)})_j = \frac{n}{2}e^{-ij\theta} = \frac{n}{2}(v^{(n-1)})_j$ and $(Av^{(n-1)})_j = \frac{n}{2}e^{ij\theta} = \frac{n}{2}(v^{(1)})_j$, so that $A(v^{(1)} \pm v^{(n-1)}) = \pm\frac{n}{2}(v^{(1)} \pm v^{(n-1)})$. Thus $\{v^{(0)}, v^{(2)}, v^{(3)}, \dots, v^{(n-2)}, v^{(1)} + v^{(n-1)}, v^{(1)} - v^{(n-1)}\}$ is a basis for $\mathbb{C}^n$ of eigenvectors of A with the claimed eigenvalues.

Finally, the determinant of $I + A$ is the product of $(1 + \lambda)$ over all eigenvalues λ of A; in this case, $\det(I + A) = (1 + n/2)(1 - n/2) = 1 - n^2/4$.

B–6 Choose a sequence $p_1, p_2, \dots$ of primes as follows. Let p_1 be any prime dividing an element of S. To define p_{j+1} given $p_1, \dots, p_j$, choose an integer $N_j \in S$ relatively prime to $p_1 \cdots p_j$ and let p_{j+1} be a prime divisor of N_j, or stop if no such N_j exists.

Since S is finite, the above algorithm eventually terminates in a finite sequence $p_1, \dots, p_k$. Let m be the smallest integer such that $p_1 \cdots p_m$ has a divisor in S. (By the assumption on S with $n = p_1 \cdots p_k$, $m = k$ has this property, so m is well-defined.) If $m = 1$, then $p_1 \in S$, and we are done, so assume $m \geq 2$. Any divisor d of $p_1 \cdots p_m$ in S must be a multiple of p_m, or else it would also be a divisor of $p_1 \cdots p_{m-1}$, contradicting the choice of m. But now $\gcd(d, N_{m-1}) = p_m$, as desired.